CS103
Fall 2025

Lecture 01:

Mathematical Proofs

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Precise, clearly-articulated arguments.

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Mathematical Proofs

Precise, clearly-articulated arguments.

Today's Lecture

Outline

How to Write a Proof

- Synthesizing definitions, intuitions, and conventions.

Proofs on Numbers

- Working with odd and even numbers.

Universal and Existential Statements

- Two important classes of statements.

Variable Ownership

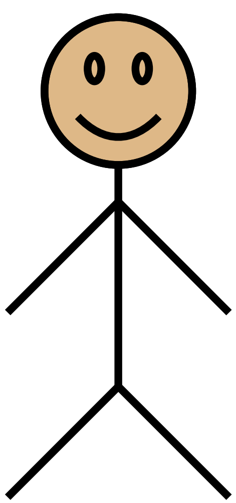
- Who owns what?

To kick things off:

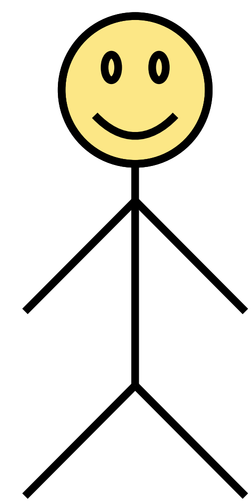
What is a proof?

Proof as Dialog

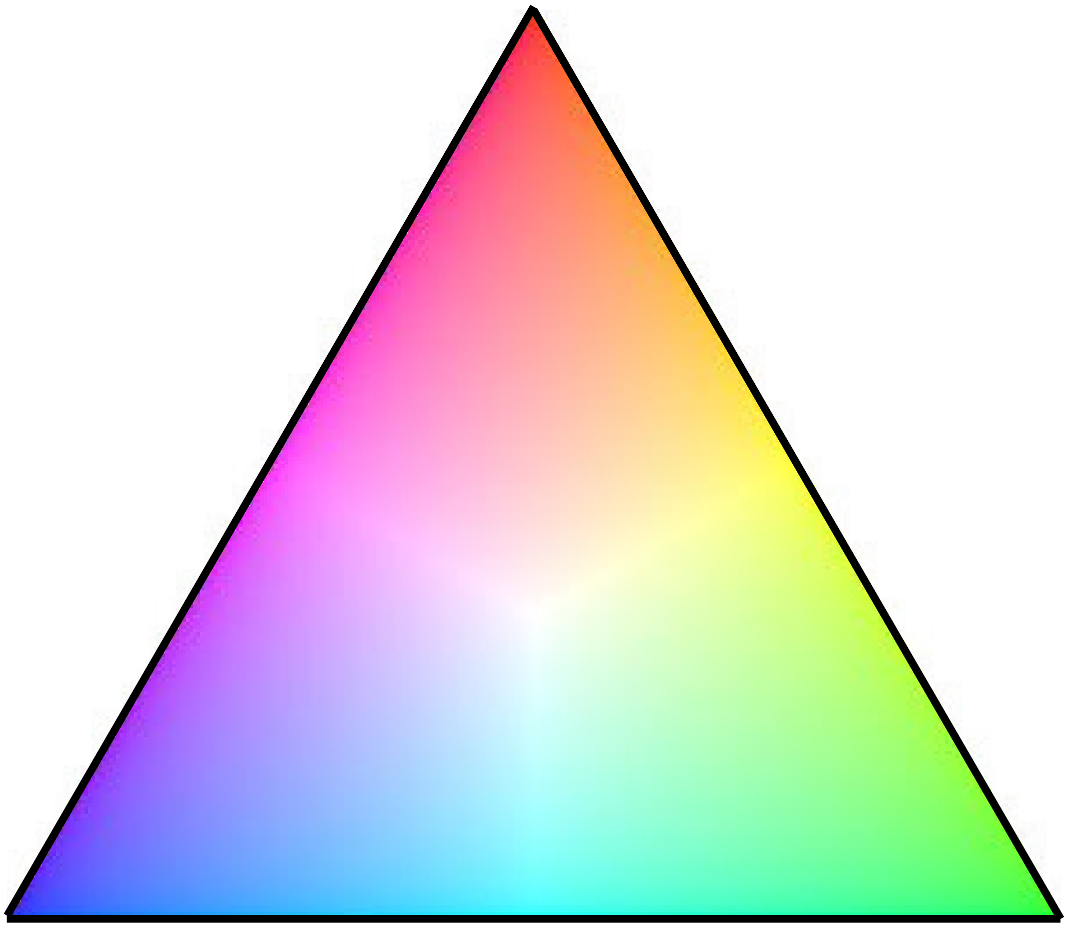
- A mathematical proof is a dialog between two parties: a ***proof writer*** and a ***proof reader***.
  - The ***proof writer*** knows a mathematical fact.
  - The ***proof reader*** is honest but skeptical.
- The proof writer's job is to take the reader on a journey from ignorance to understanding.



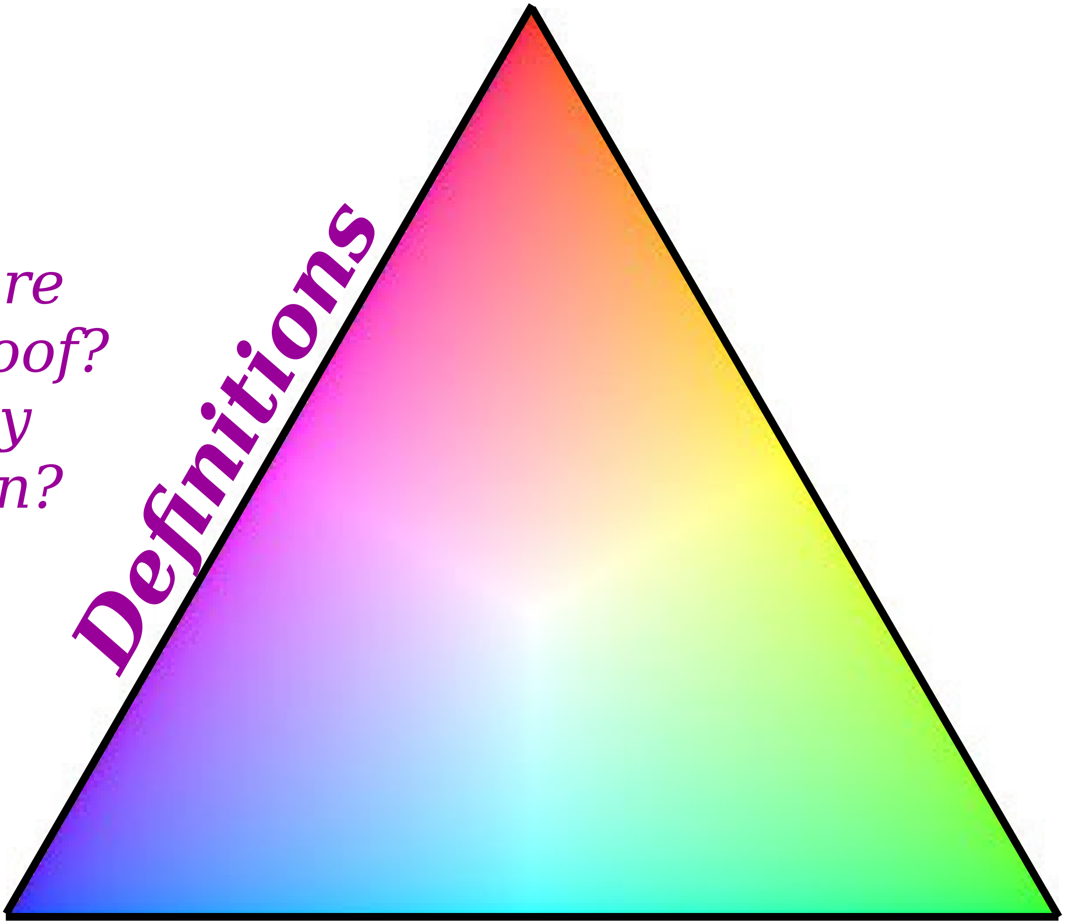
Proof Writer (You)



Proof Reader



What terms are used in this proof? What do they formally mean?



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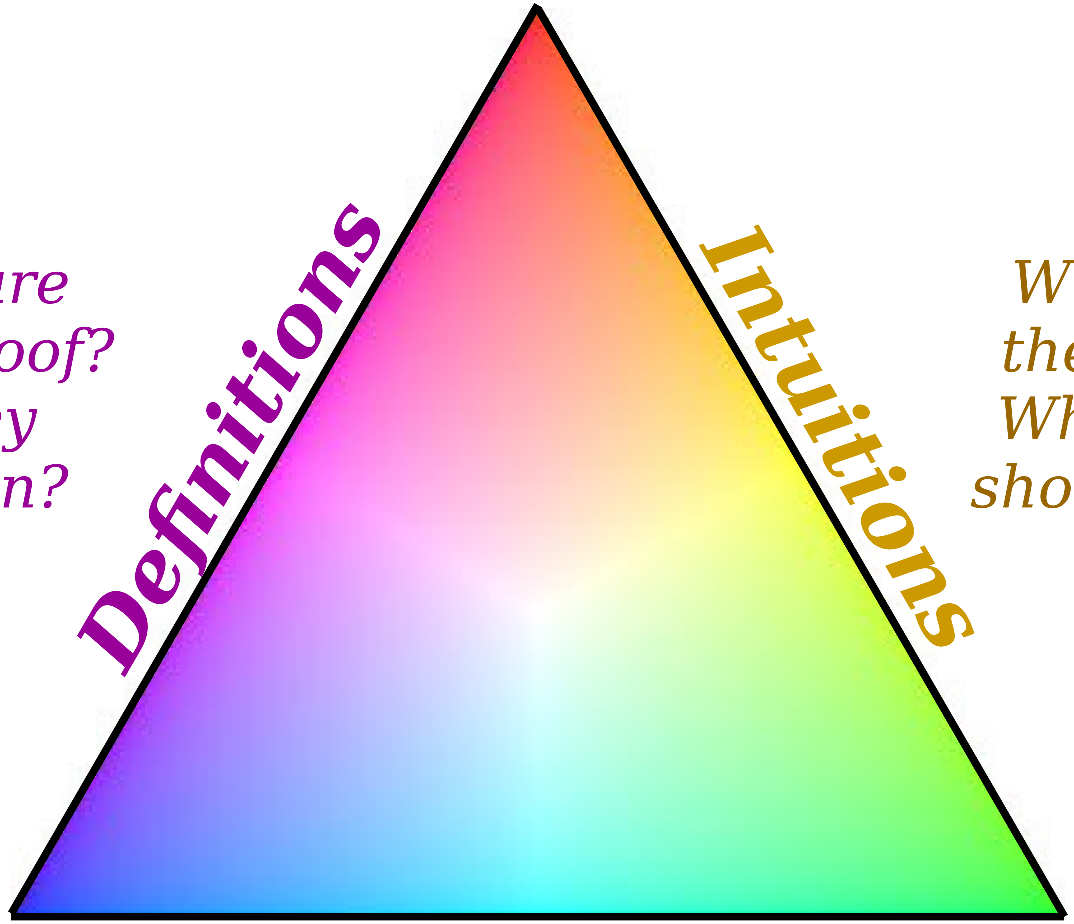
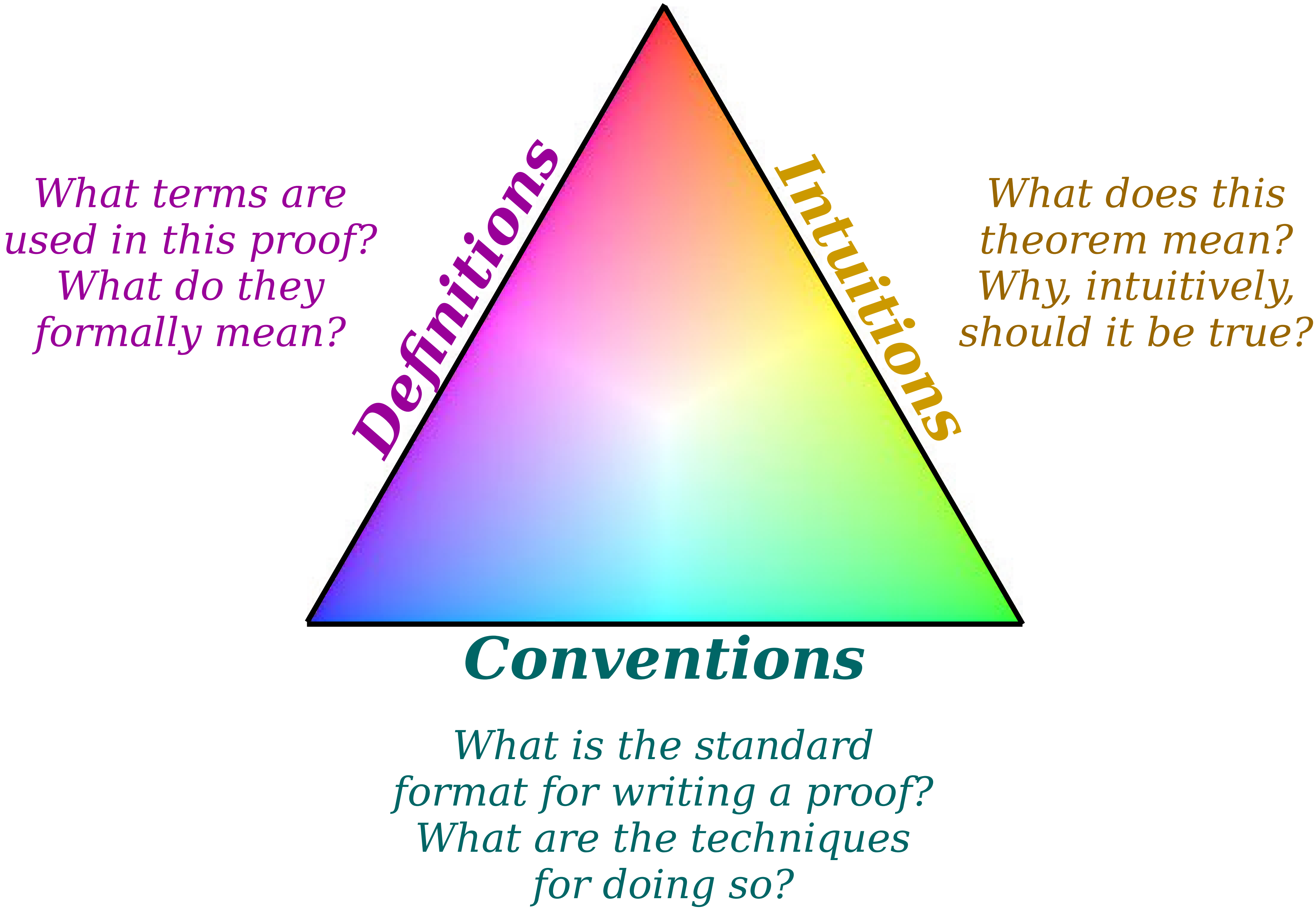


What does this theorem mean? Why, intuitively, should it be true?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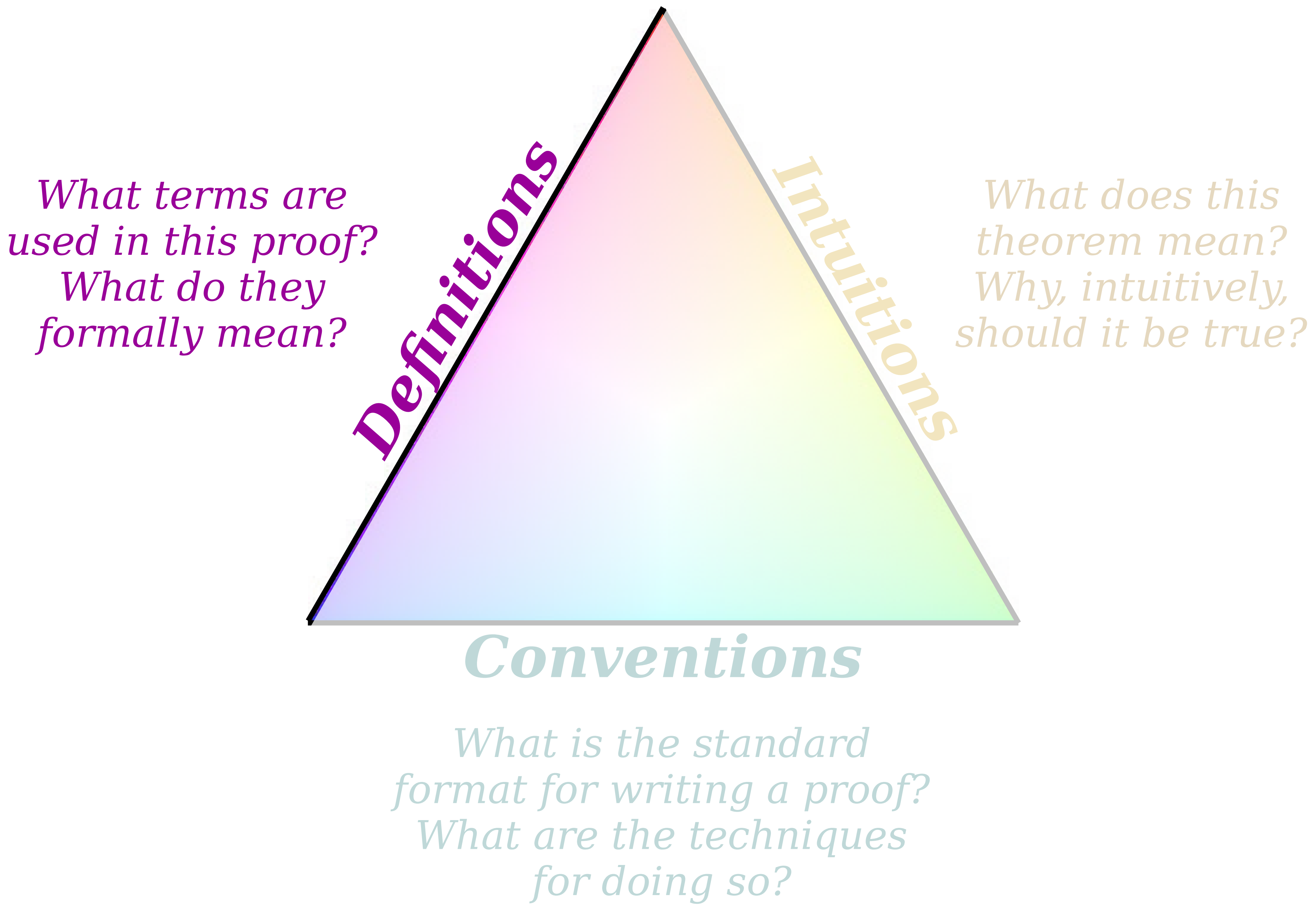
Writing our First Proof

Theorem: If n is an even integer, then n^2 is even.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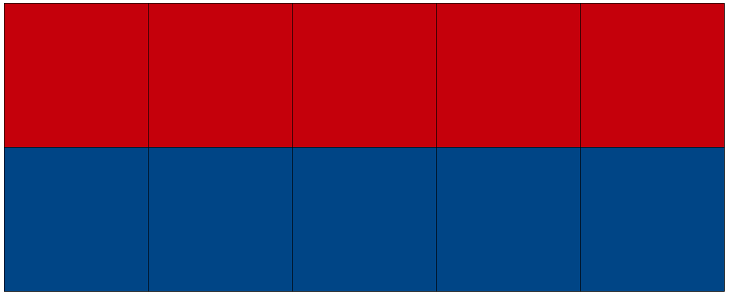
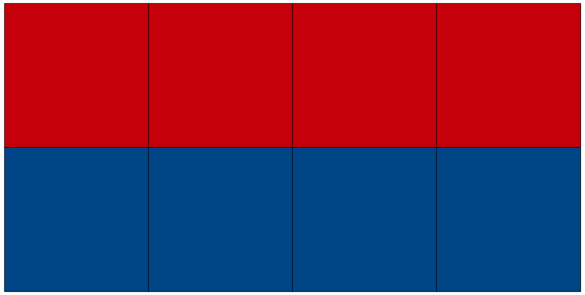


$2 \cdot \mathbf{5}$

8



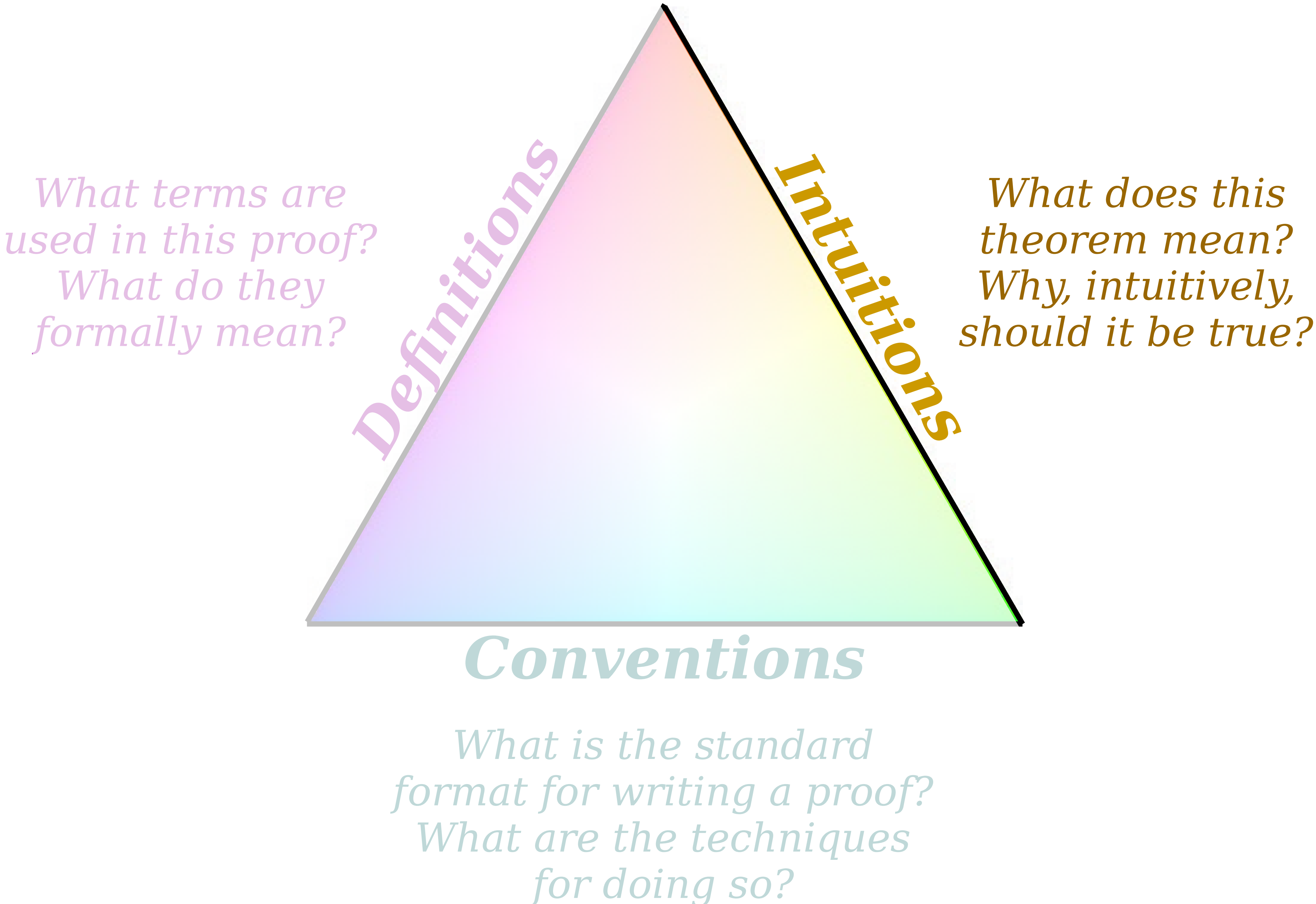
$2 \cdot \mathbf{4}$

0

$2 \cdot \mathbf{0}$

An integer n is called ***even*** if there is an integer k where $n = 2k$.

Theorem: If n is an even integer, then n^2 is even.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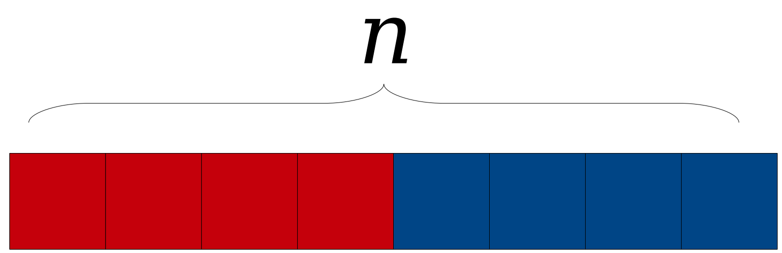
Let’s Try Some Examples!

Theorem: If n is an even integer, then n^2 is even.

Let's Draw Some Pictures!



$n = 2k$

Theorem: If n is an even integer, then n^2 is even.

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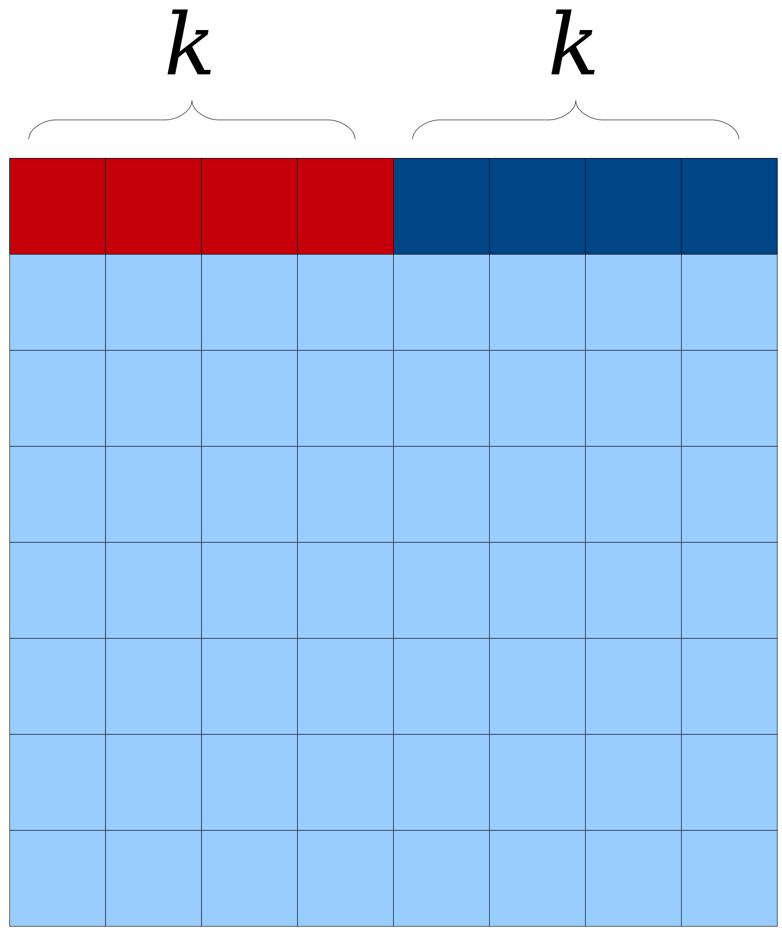

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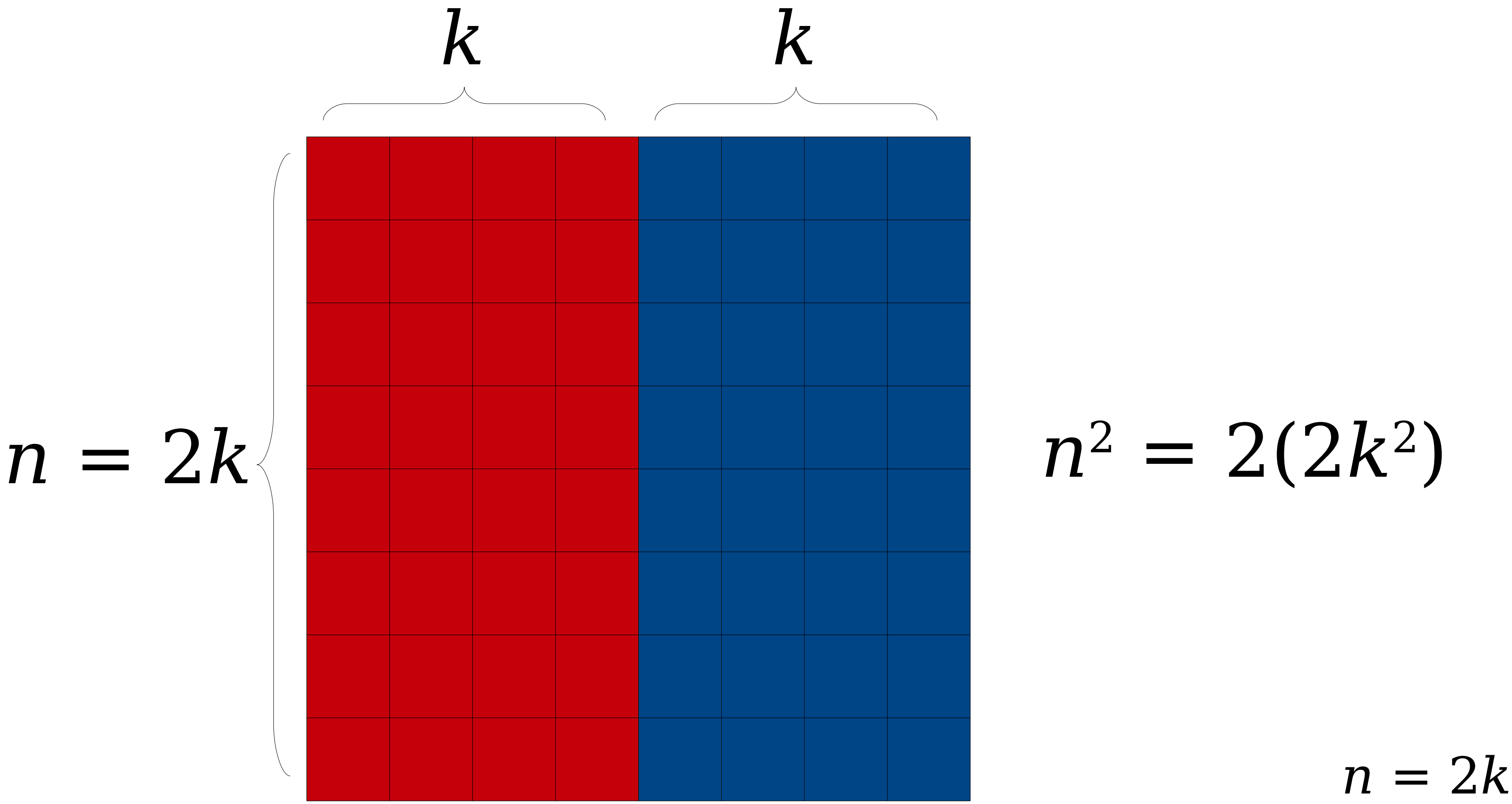

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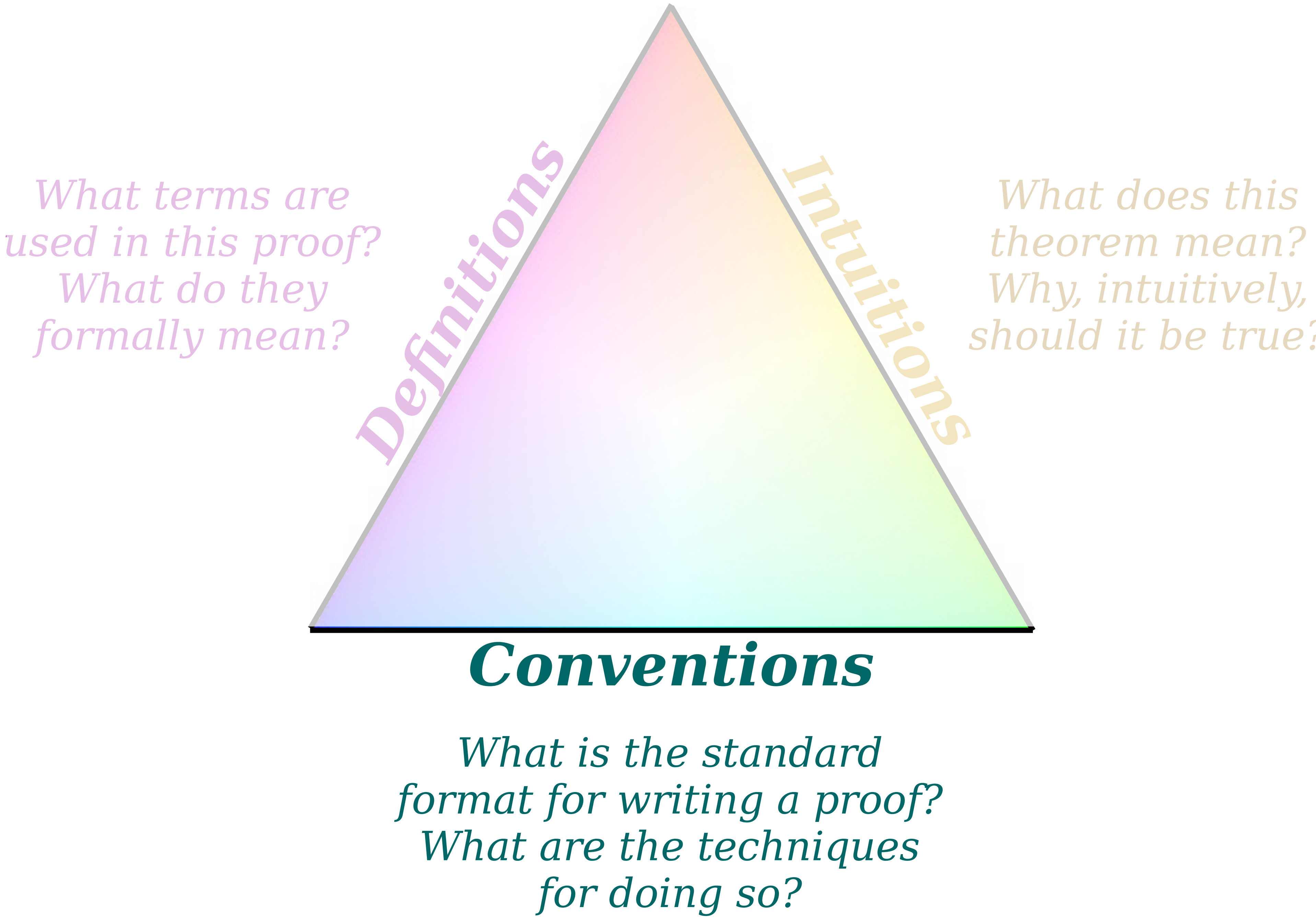
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Proof: Assume n is an even integer. We want to show that n^2 is even.

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$$n^2 = (2k)^2$$

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> This symbol means "end of proof"

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start by asking the reader to assume that ***P*** is true.

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To prove a statement of the form

"If *P* is true, then *Q* is true,"

we assume ***P*** is true, then need to show that ***Q*** is true. Here, we're telling the reader where we're headed.

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We apply the definition of an even integer. We need to use this definition to make this proof rigorous.

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Notice how we use the value of ***k*** that we obtained above. Giving names to quantities, allows us to manipulate them. This is similar to variables in programs.

Our First Proof!

Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer. We want to show that n^2 is even.

Since ... h that n ...

Our ultimate goal is to prove that n^2 is even. This means that we need to find some m where $n^2 = 2m$. Here, we're explicitly showing how we can do that.

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$$
\begin{aligned}
n^2 &= (2k)^2 \\
&= 4k^2 \\
&= 2(2k^2)
\end{aligned}
$$

Hey, that's what we said we were going to do! We're done.

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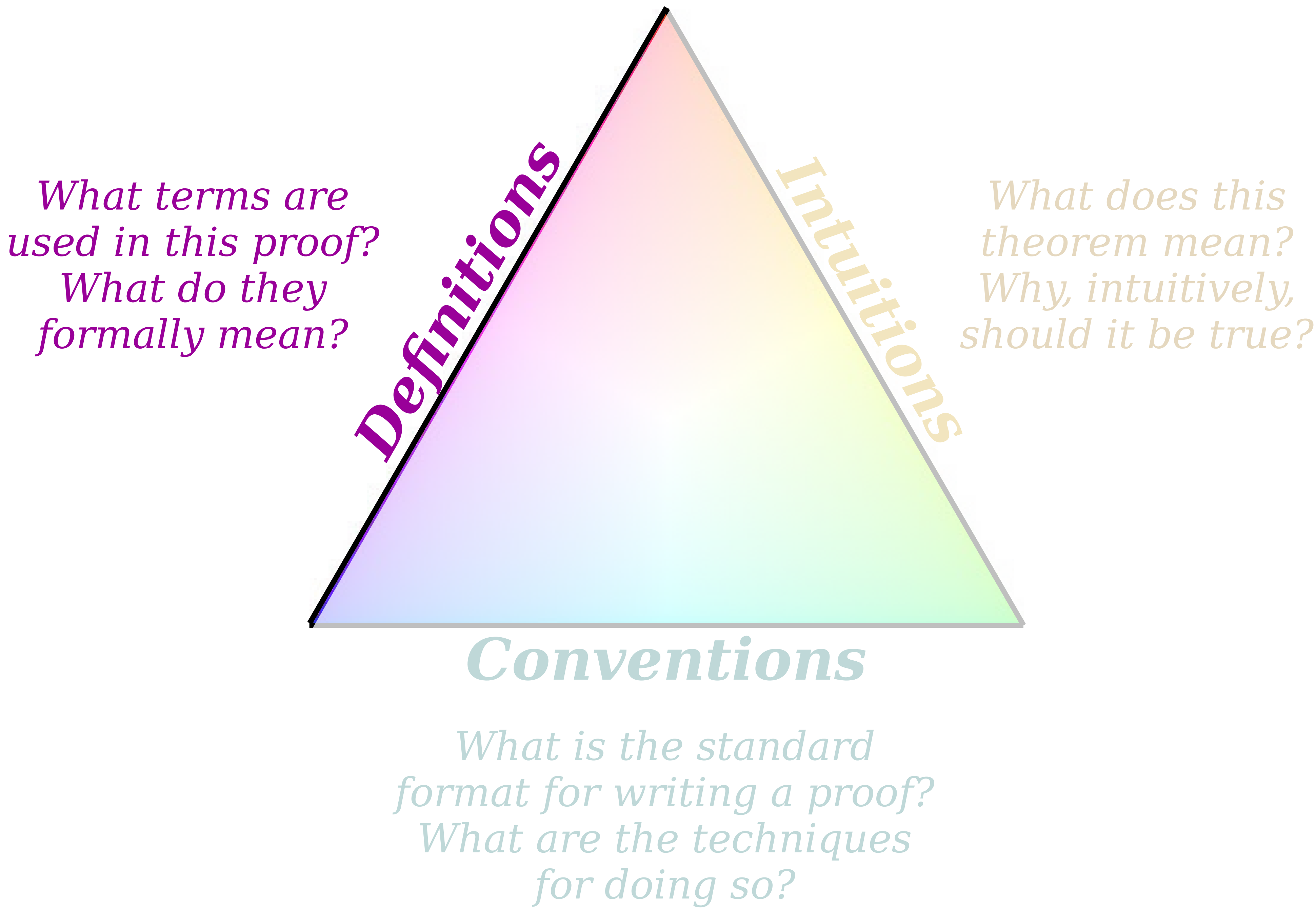
Our Next Proof

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.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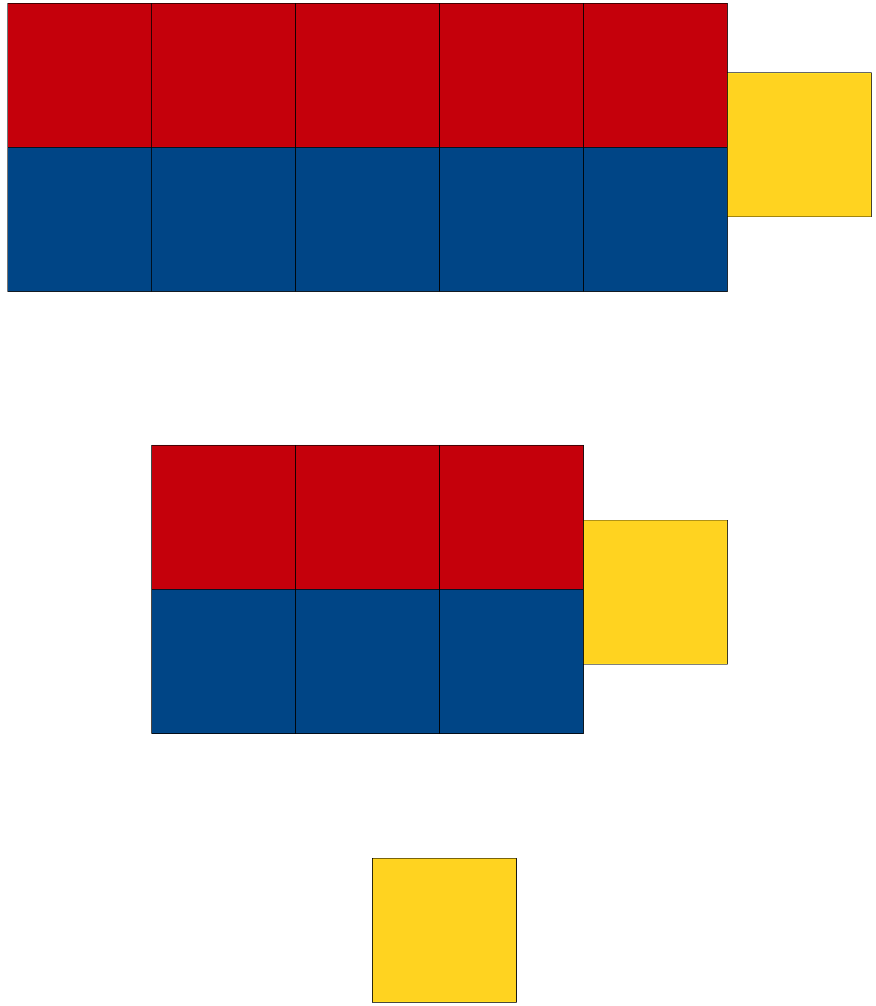
11 — $2 \cdot \mathbf{5} + \mathbf{1}$

7 — $2 \cdot \mathbf{3} + \mathbf{1}$

1 — $2 \cdot \mathbf{0} + \mathbf{1}$

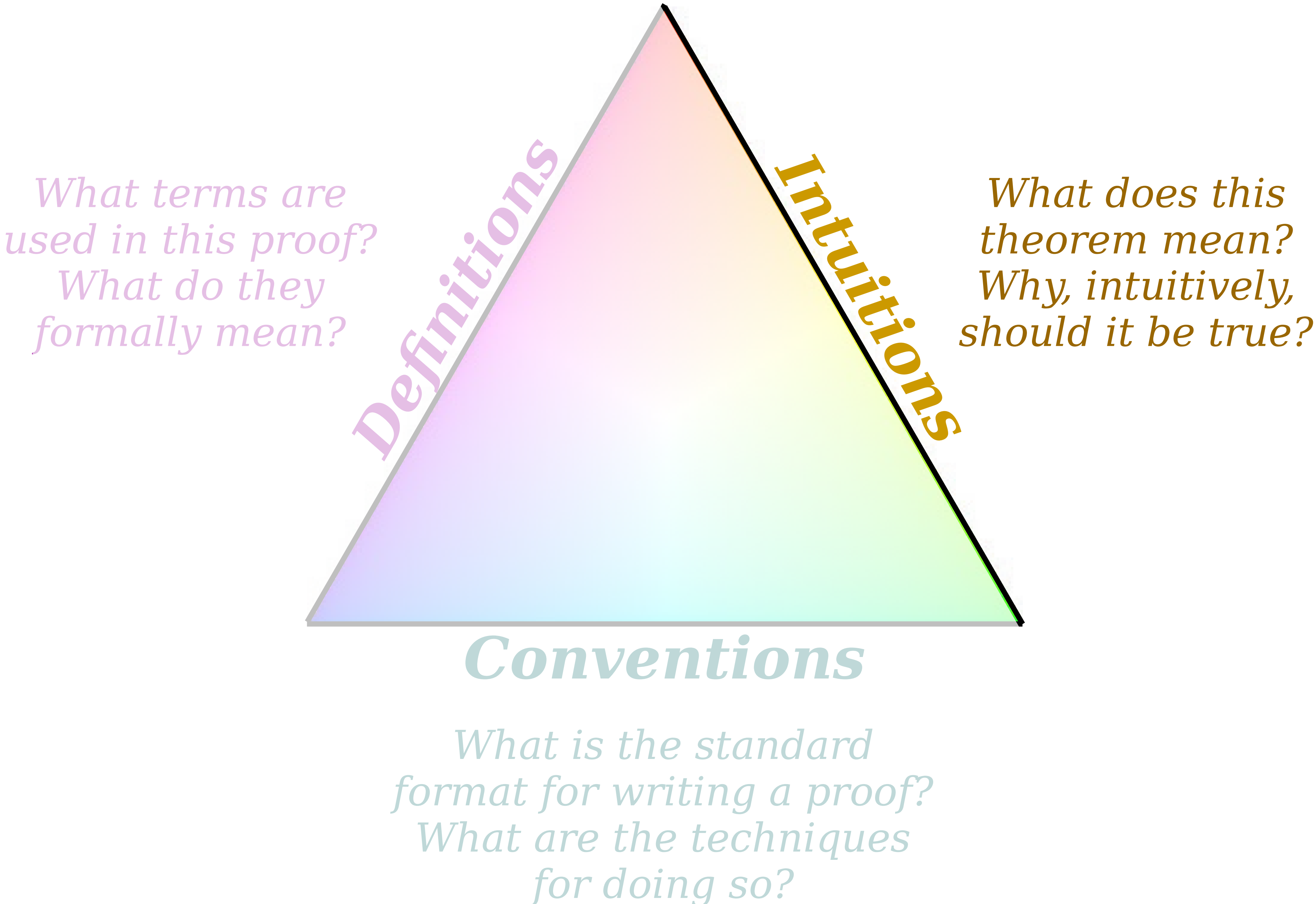


An integer n is called ***odd*** if
there is an integer k where $n = 2k+1$.

Going forward, we'll assume the following:

1. Every integer is either even or odd.
2. No integer is both even and odd.

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.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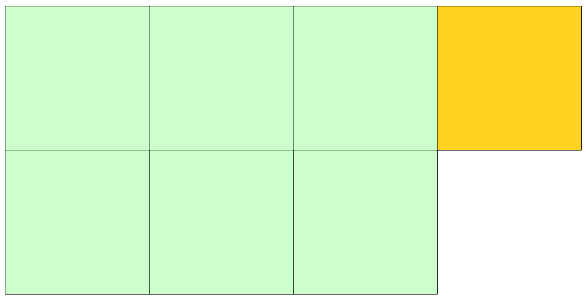

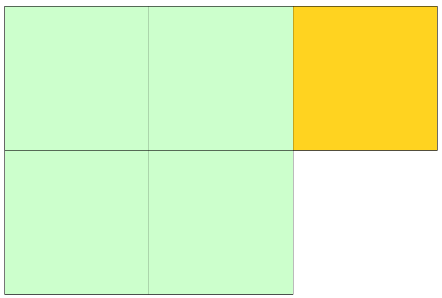

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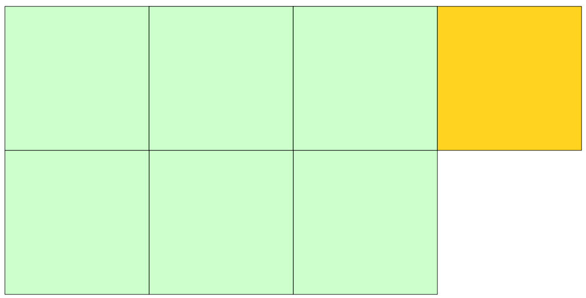

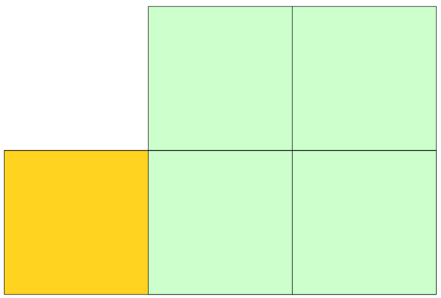

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Let’s Do Some Math!

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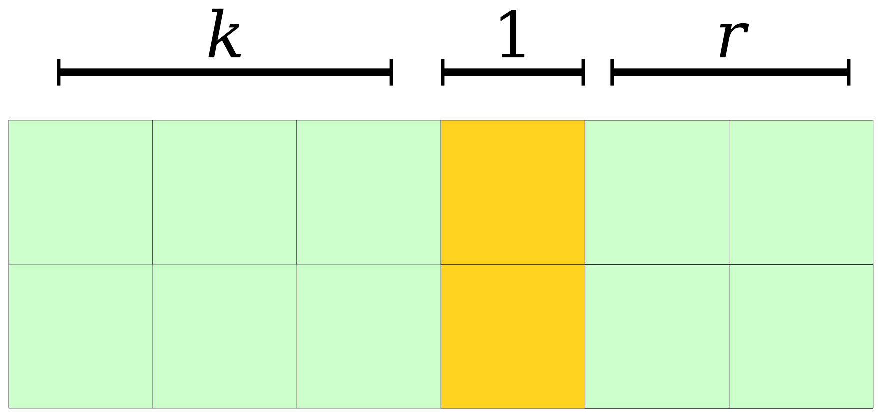
Let's Do Some Math!

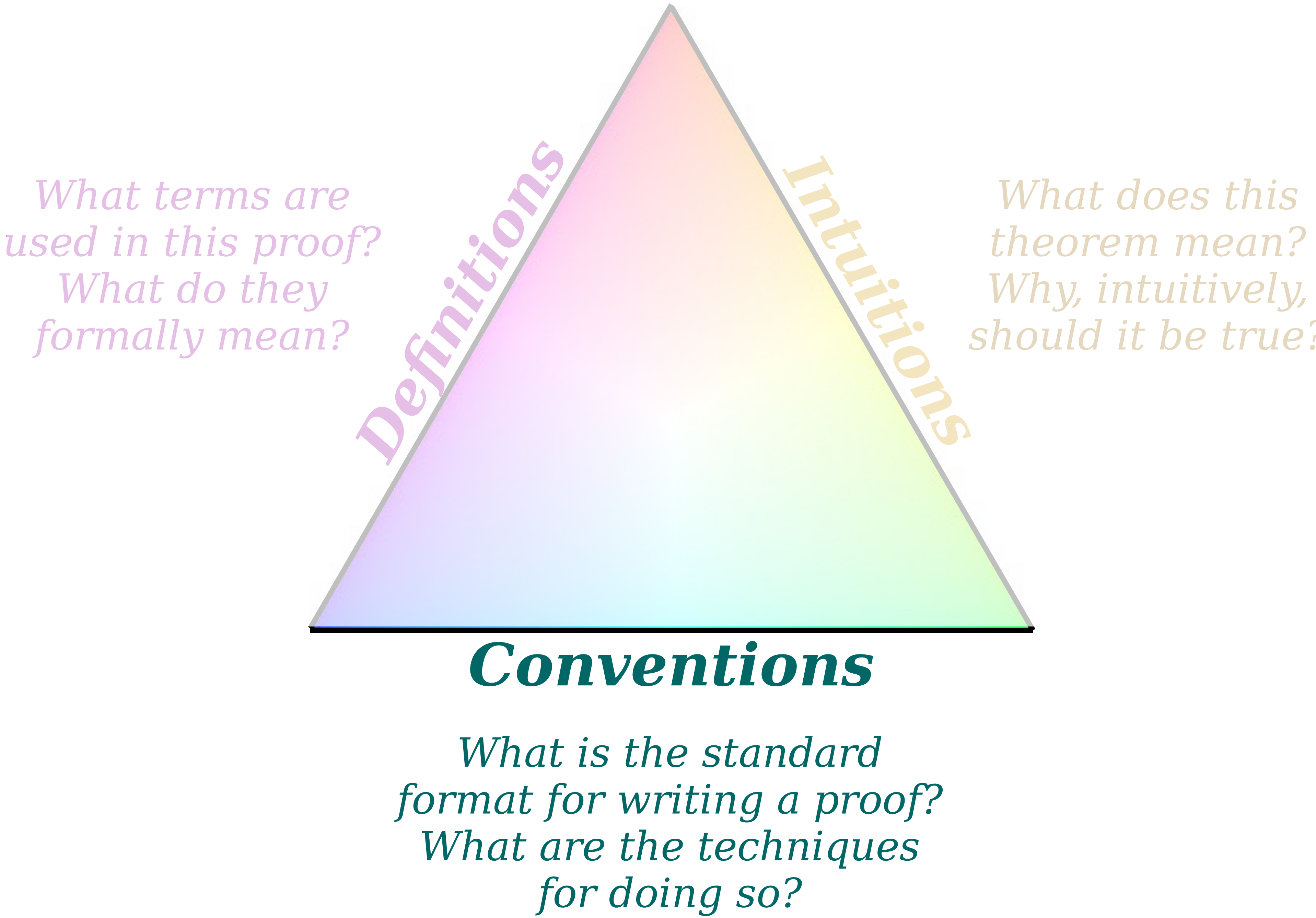
$2k+1$



$2r+1$

$$(2k+1) + (2r+1) = 2(k + r + 1)$$

Theorem: For any integers m and n, if m and n are odd, then $m+n$ is even.



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Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \tag{1}$$

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Proof: Consider any arbitrary integers m and n where m and n are odd. We need to show that $m + n$ is even.

Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \quad (1)$$

Similarly, because n is odd there must be some integer r such that

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$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \end{aligned}$$

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We ask the reader to make an *arbitrary choice*. Rather than specifying what ***m*** and ***n*** are, we're signaling to the reader that they could, in principle, supply any choices of ***m*** and ***n*** that they'd like.

By letting the reader pick ***m*** and ***n*** arbitrarily, anything we prove about ***m*** and ***n*** will generalize to all possible choices for those values.

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Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers m and n where m and n are odd. We need to show that $m + n$ is even.

Since m i

Similarly, ch that

By adding

To prove a statement of the form

"If *P* is true, then *Q* is true,"

after assuming ***P*** is true, you need to show that ***Q*** is true.

$$= 2k + 2r + 2$$

$$= 2(k + r + 1). \qquad (3)$$

Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Numbering these equalities lets us refer back to them later on, making the flow of the proof a bit easier to understand.

Proof: Consider any
odd. We need to s

Since m is odd, w

$$m = 2k + 1. \qquad (1)$$

Similarly, because n is odd there must be some integer r such that

$$n = 2r + 1. \qquad (2)$$

By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \qquad (3) \end{aligned}$$

Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

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(2)

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… $+ 2$

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…nteger s (namely, $k + r + 1$)

…see that $m + n$ is even, as

This is a complete sentence! Proofs are expected to be written in complete sentences, so you'll often use punctuation at the end of formulas.

We recommend using the "mugga mugga" test – if you read a proof and replace all the mathematical notation with "mugga mugga," what comes back should be a valid sentence.

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers m and n where m and n are odd. We need to show that $m + n$ is even.

Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \tag{1}$$

Similarly, because n is odd there must be some integer r such that

$$n = 2r + 1. \tag{2}$$

By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \end{aligned} \tag{3}$$

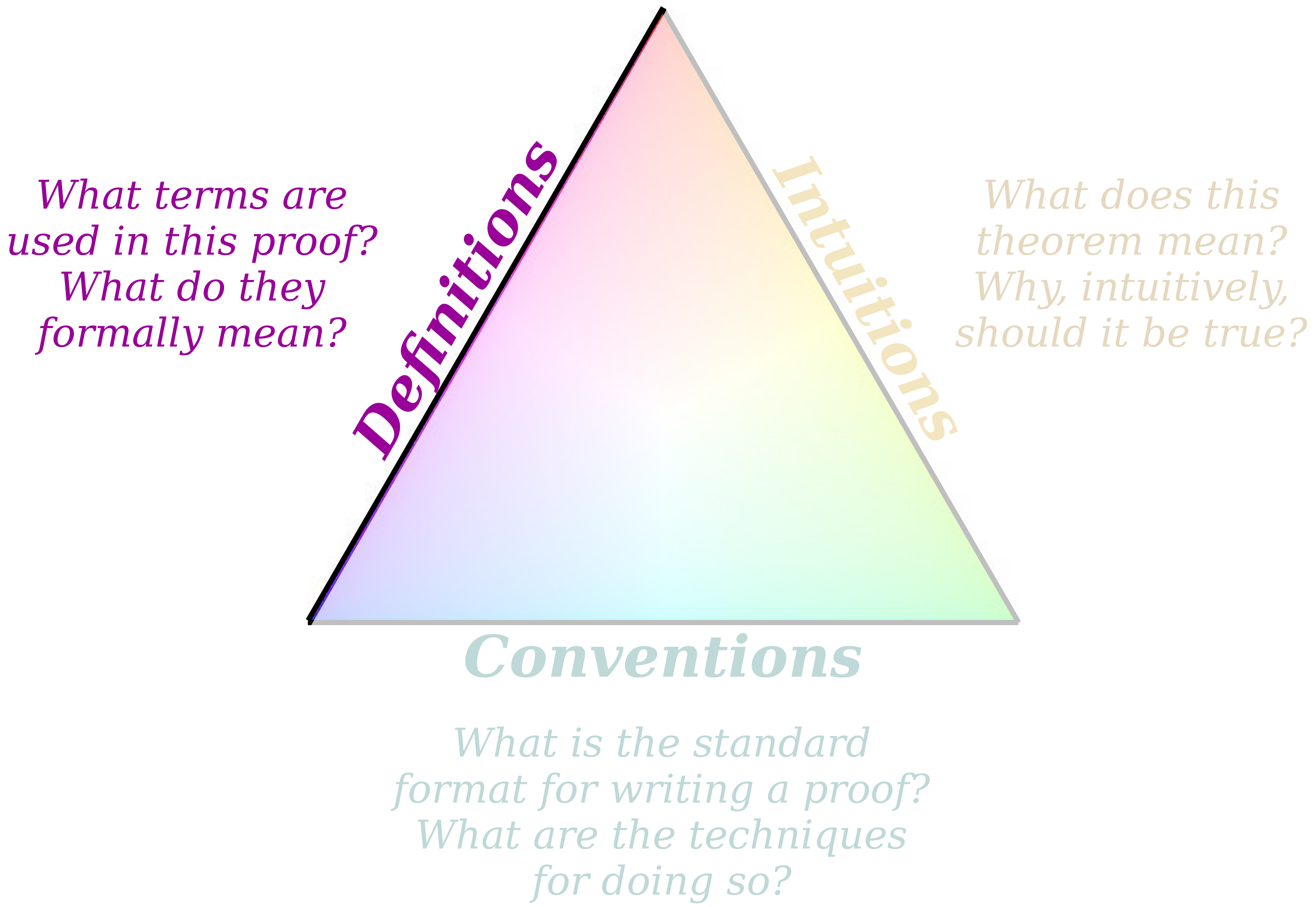
Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

Some Little Exercises

- Here's a list of other theorems that are true about odd and even numbers:
  - ***Theorem:*** The sum and difference of any two even numbers is even.
  - ***Theorem:*** The sum and difference of an odd number and an even number is odd.
  - ***Theorem:*** The product of any integer and an even number is even.
  - ***Theorem:*** The product of any two odd numbers is odd.
- Going forward, we'll just take these results for granted. Feel free to use them in the problem sets.
- If you'd like to practice the techniques from today, try your hand at proving these results!

Universal and Existential Statements

Theorem: For any odd integer n,
there exist integers r and s where $r^2 - s^2 = n$.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Theorem: For any odd integer n,
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Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

This result is true for every possible choice of odd integer n. It'll work for $n = 1$, $n = 137$, $n = 103$, etc.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

We aren't saying this is true for every choice of r and s. Rather, we're saying that *somewhere out there* are choices of r and s where this works.

Universal vs. Existential Statements

- A ***universally-quantified statement*** is a statement of the form

  For all *x*, `[some-property]` holds for *x*.

- We've seen how to prove these statements.
- An ***existentially-quantified statement*** is a statement of the form

  There is an *x* where `[some-property]` holds for *x*.

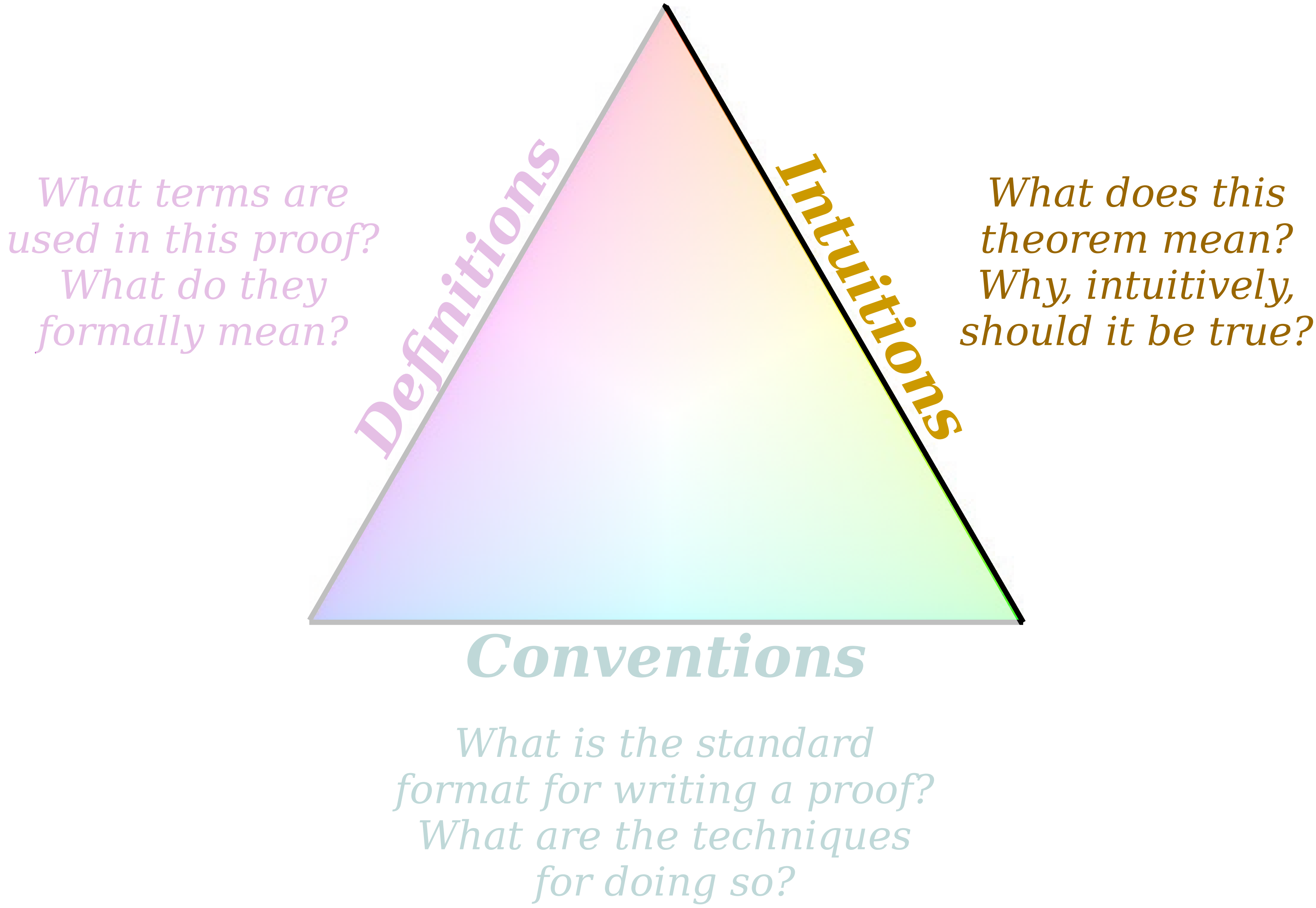
- How do you prove an existentially-quantified statement?

Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existentially-quantified statement of the form

There is an *x* where `[some-property]` holds for *x*.

- ***Simplest approach:*** Search far and wide, find an *x* that has the right property, then show why your choice is correct.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Let's Try Some Examples!

$1 = \_\_\_^2 - \_\_\_^2$

$3 = \_\_\_^2 - \_\_\_^2$

$5 = \_\_\_^2 - \_\_\_^2$

$7 = \_\_\_^2 - \_\_\_^2$

$9 = \_\_\_^2 - \_\_\_^2$

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Let's Try Some Examples!

$$1 = \mathbf{1}^2 - \mathbf{0}^2$$

$$3 = \mathbf{2}^2 - \mathbf{1}^2$$

$$5 = \mathbf{3}^2 - \mathbf{2}^2$$

$$7 = \mathbf{4}^2 - \mathbf{3}^2$$

$$9 = \mathbf{5}^2 - \mathbf{4}^2$$

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

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$$1 = 2 \cdot \_ + 1 = \mathbf{1}^2 - \mathbf{0}^2$$

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$$\mathbf{3} + 1 = \mathbf{4}^2 - \mathbf{3}^2$$

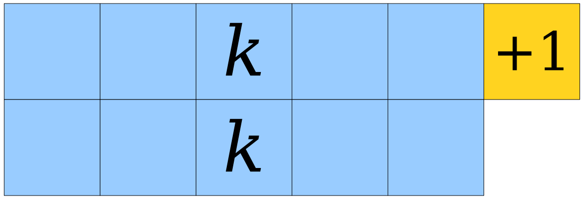
$$\mathbf{4} + 1 = \mathbf{5}^2 - \mathbf{4}^2$$

Educated Guess:

$2k + 1 = (k+1)^2 - k^2$.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Let’s Draw Some Pictures!



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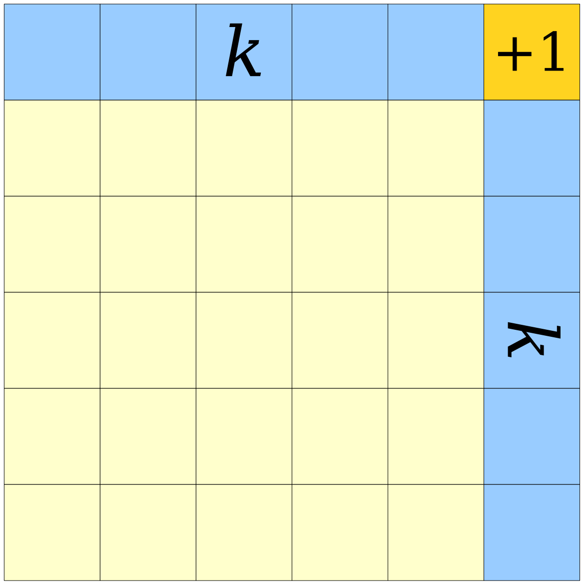
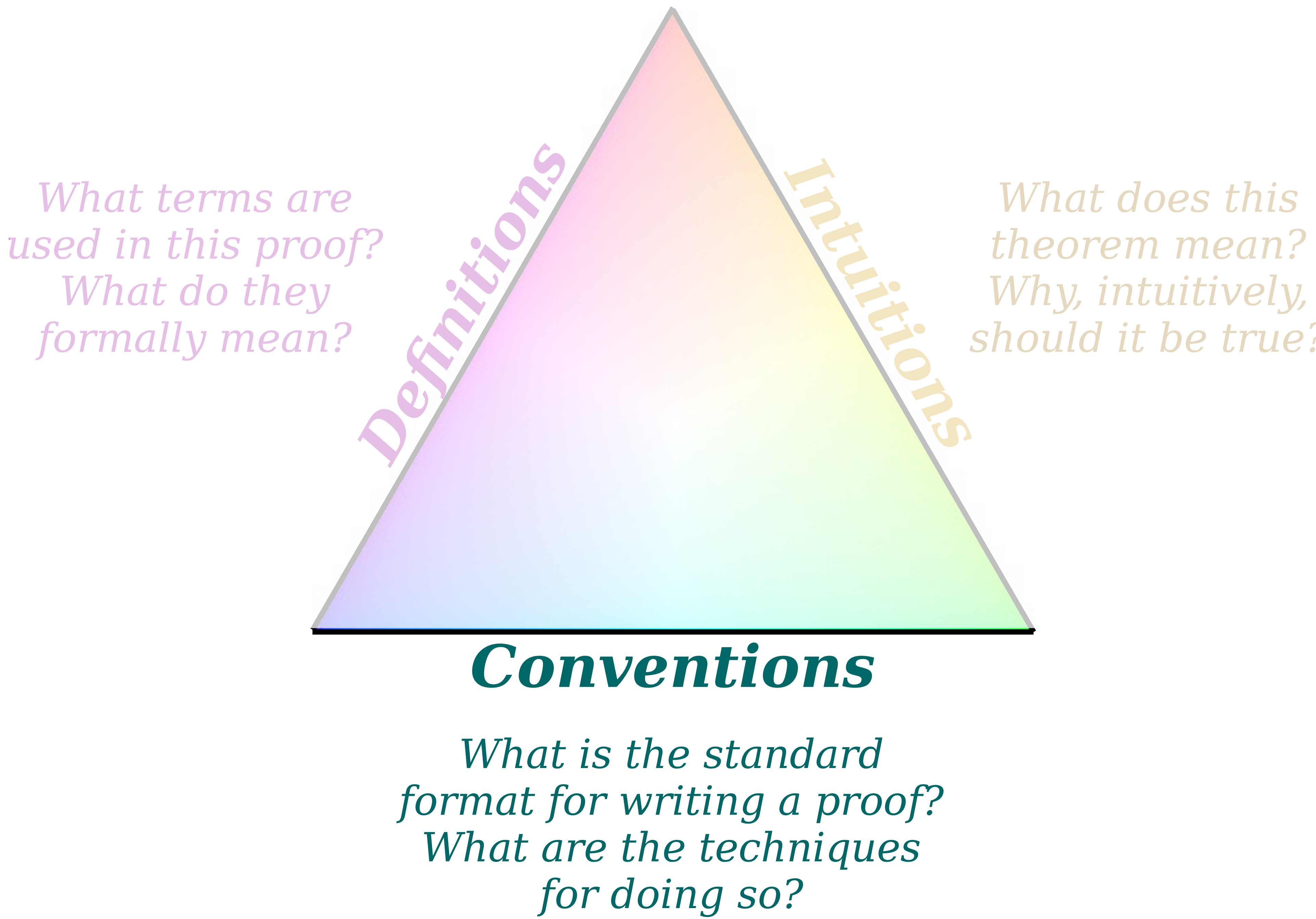


$$(k+1)^2 - k^2 = 2k+1$$

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.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$$r^2 - s^2 \quad = \quad (k+1)^2 - k^2$$

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This means that $r^2 - s^2 = n$, which is what we needed to show.

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Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since r … here $n = 2k$ … we see that

> We ask the reader to make an *arbitrary choice*. Rather than specifying what ***n*** is, we're signaling to the reader that they could, in principle, supply any choice ***n*** that they'd like.

$$= \quad 2k + 1$$

$$= \quad n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

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$$
\begin{aligned}
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&= k^2 + 2k + 1 - k^2 \\
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&= n.
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$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

> As always, it's helpful to write out what we need to demonstrate with the rest of the proof.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$r^2 - s^2 = (k+1)^2$$

$$= k^2 + 2$$

$$= 2k + 1$$

$$= n.$$

This means that $r^2 - s^2 = n$, w
to show. ■

We're trying to prove an existential statement. The easiest way to do that is to just give concrete choices of the objects being sought out.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

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Check the appendix to this slide deck for more about who gets to choose values.

Time-Out for Announcements!

Working in Pairs

- Starting with Problem Set One, you are allowed to work either individually or in pairs.
  - Each pair should make a single joint submission.
- We have advice about how to work effectively in pairs up on the course website – check the "Guide to Partners."
- Want to work in a pair, but don't know who to work with? Fill out ***this Google form*** and we'll connect you with a partner on Friday.

Problem Set 0

- Problem Set 0 is due this ***Friday*** at ***1:00PM***.
  - (It needs to be completed individually.)
- Need help getting Qt Creator installed? There's a Qt Creator help session running ***tomorrow***, ***7PM – 9PM***, in ***CoDa B45***.
  - We recommend installing Qt Creator by this evening so that if you run into trouble, you can stop by this help session.

CS103 ACE

- Reminder: There’s an optional companion course, CS103 ACE, that runs in parallel with CS103.
- CS103 ACE meets Thursdays 1:30 – 3:20PM and provides additional practice with the course material in a small group setting.
- This Thursday’s meeting is an informal, drop-in office hours session where you can lean more about the course.
- Interested? Apply online using ***this link***.

Outdoor Activities

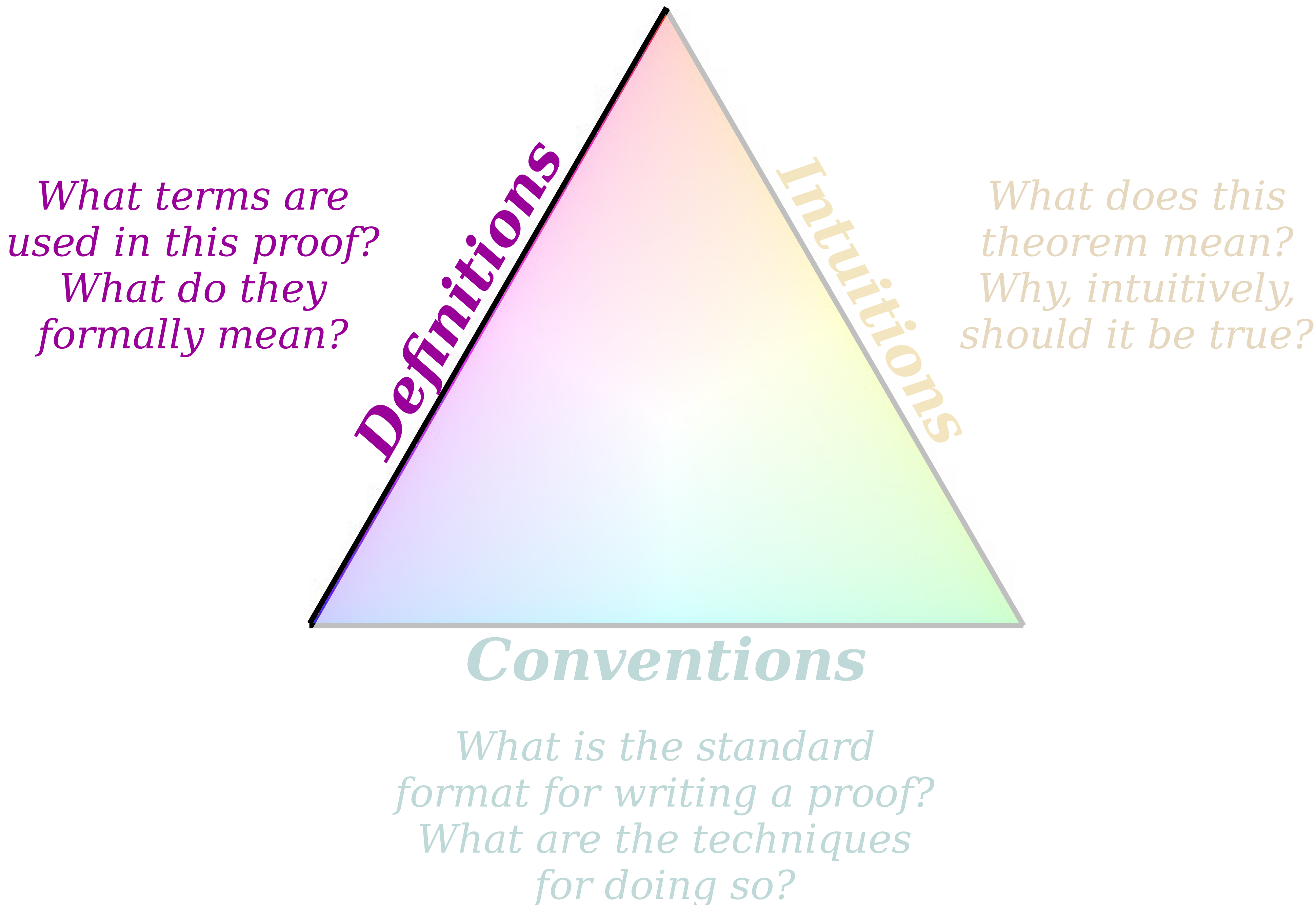
- You’re less than fifty miles from grassy mountains, redwood forests, Pacific coastline, beautiful wetlands, and more.
- Want to explore the area to see what it has to offer? Check out our (unofficial) Outdoor Activities Guide.

https://cs103.stanford.edu/outdoor_activities

- A sampler of what to check out:
  - Drive to the observatory in the mountains near San Jose and take in the views.
  - Visit a beach with an enormous colony of elephant seals.
  - Walk in redwood forests and pick your own bay leaves.
  - Grab cheap, high-quality food from unassuming strip malls.

Back to CS103!

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.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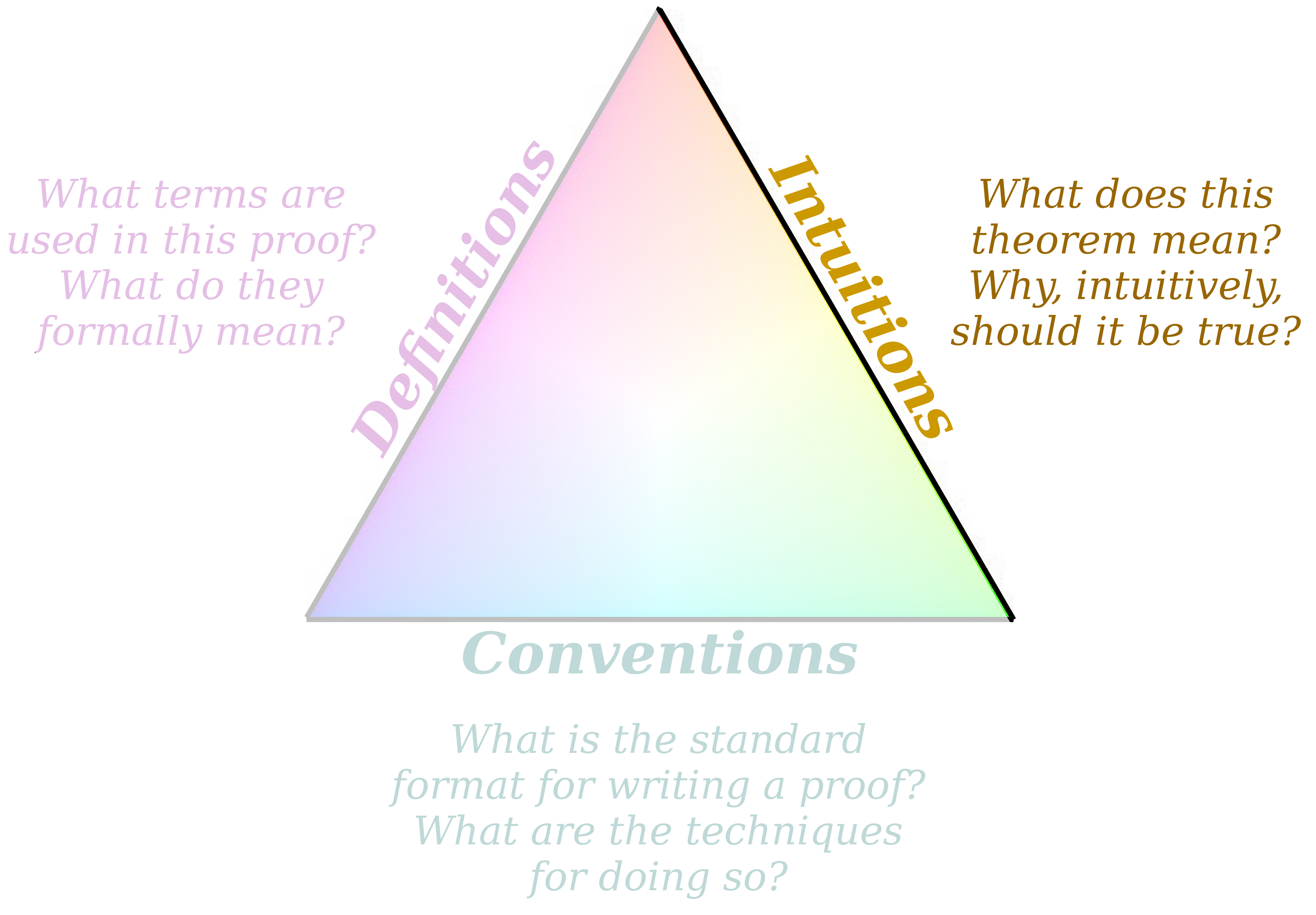
Floors and Ceilings

- The notation $\lceil x \rceil$ represents the ***ceiling*** of x, the smallest integer greater than or equal to x.
  - ***Intuition:*** Start at x on the number line, then move to the right while you're not on a tick mark.
  - What is $\lceil 1 \rceil$? What's $\lceil 1.2 \rceil$? What's $\lceil -1.2 \rceil$?
- The notation $\lfloor x \rfloor$ represents is the ***floor*** of x, the largest integer less than or equal to x.
  - ***Intuition:*** Start at x on the number line, then move to the left while you're not on a tick mark.
  - What is $\lfloor 1 \rfloor$? What's $\lfloor 1.2 \rfloor$? What's $\lfloor -1.2 \rfloor$?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Let's Try Some Examples!

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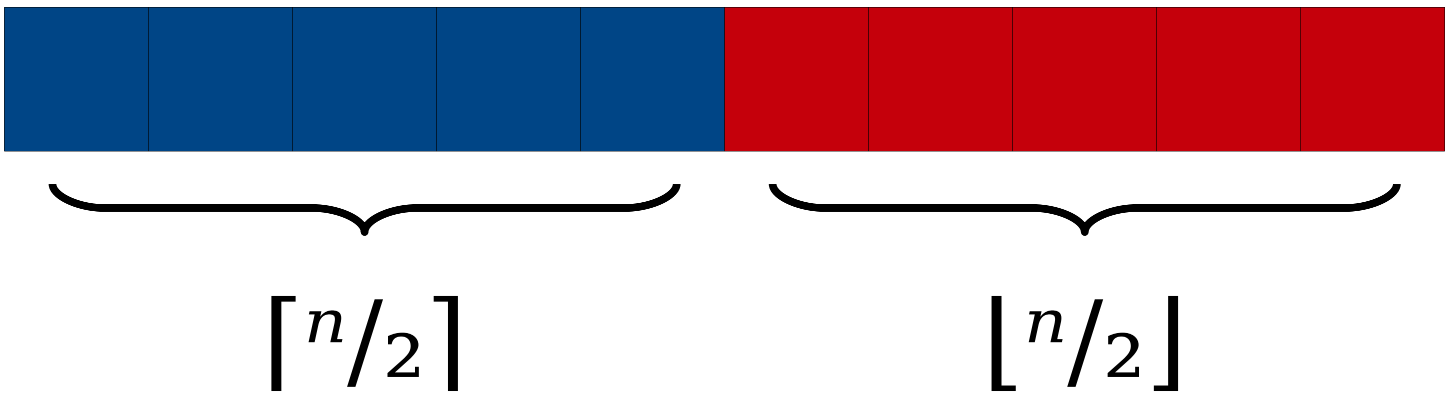
Let’s Draw Some Pictures!



$$n = 2k$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

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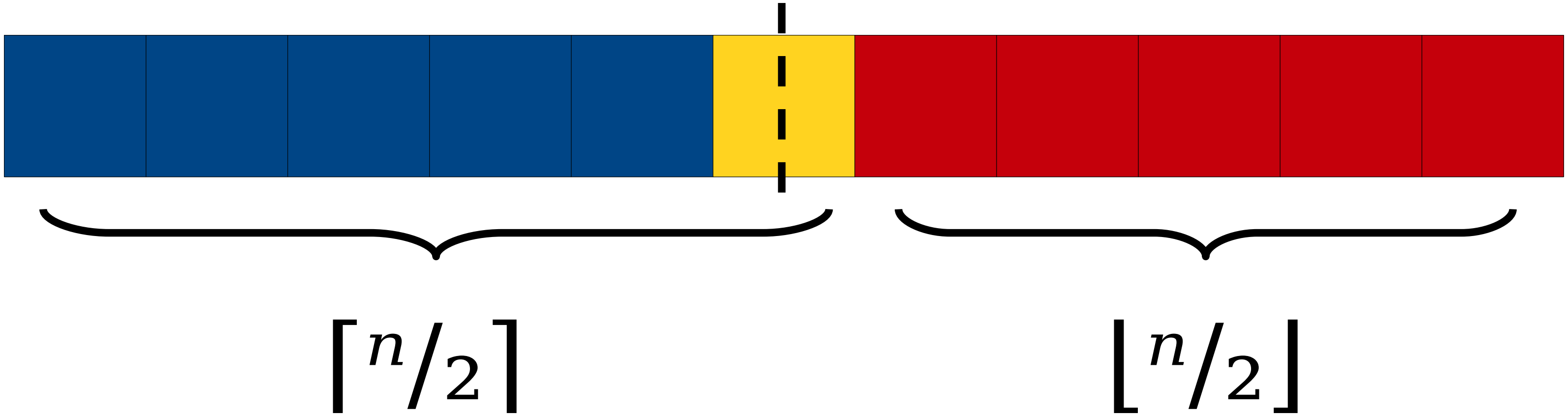
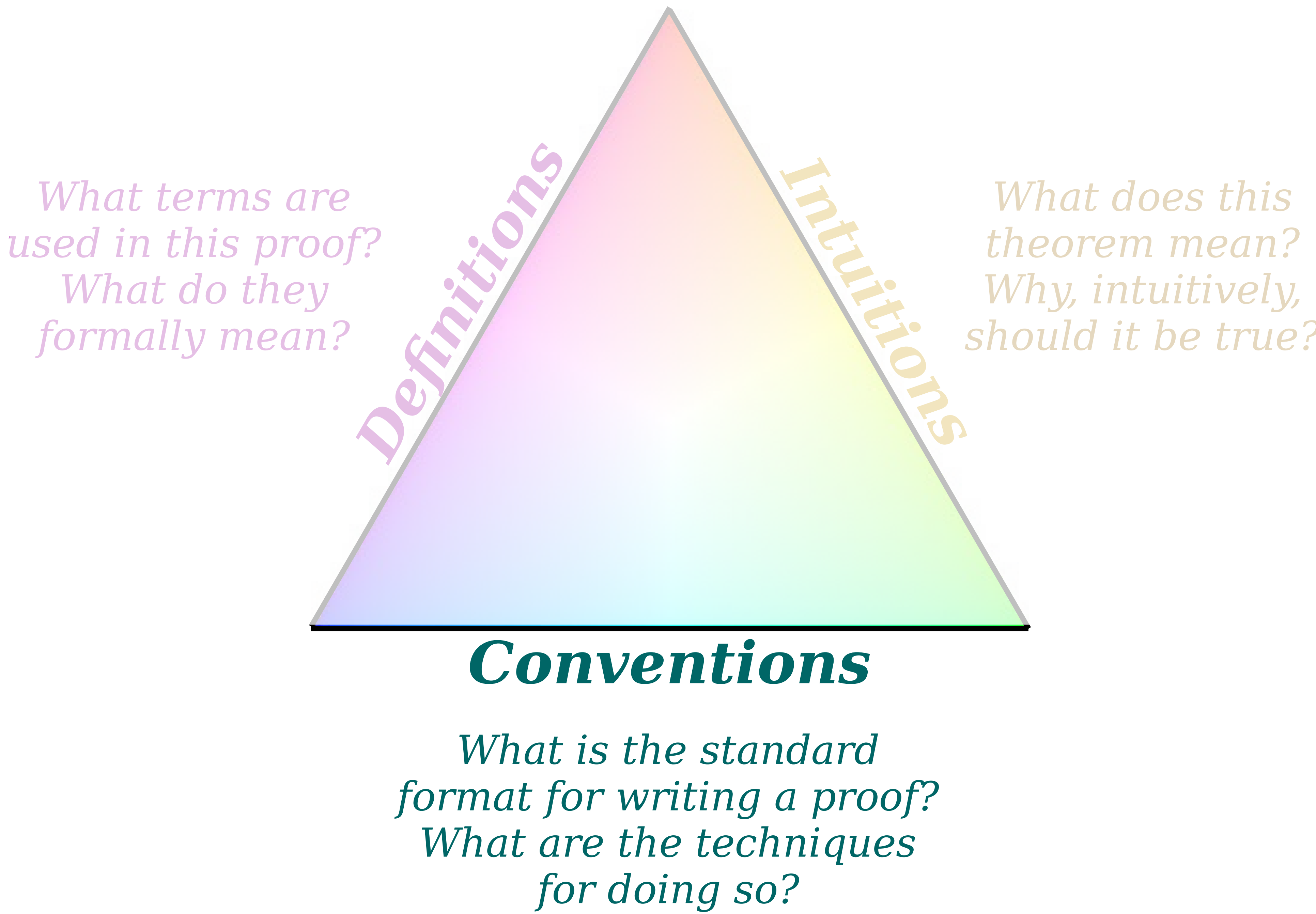


$$n = 2k + 1$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.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Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even.

Case 2: n is odd.

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even.

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This is called a **proof by cases** (or **proof by exhaustion**). We split apart into one or more cases and confirm that the result is indeed true in each of them.

(Think of it like an if/else or switch statement.)

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Case 1: n is even. This means there is an integer k such that $n = 2k$. Some algebra then tells us that

$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{2\,k}{2} \right\rceil + \left\lfloor \frac{2\,k}{2} \right\rfloor$$

Case 2: n is odd.

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$$= \lceil k \rceil + \lfloor k \rfloor$$

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$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{2k}{2} \right\rceil + \left\lfloor \frac{2k}{2} \right\rfloor$$

$$= \lceil k \rceil + \lfloor k \rfloor$$

$$= 2k$$

Case 2: n is odd.

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

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Case 2: n is odd. Then there's an integer k where $n = 2k + 1$, and

$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{2k+1}{2} \right\rceil + \left\lfloor \frac{2k+1}{2} \right\rfloor$$

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In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required.

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At the end of a split into cases, it's a nice courtesy to explain to the reader what it was that you established in each case.

$$= n.$$

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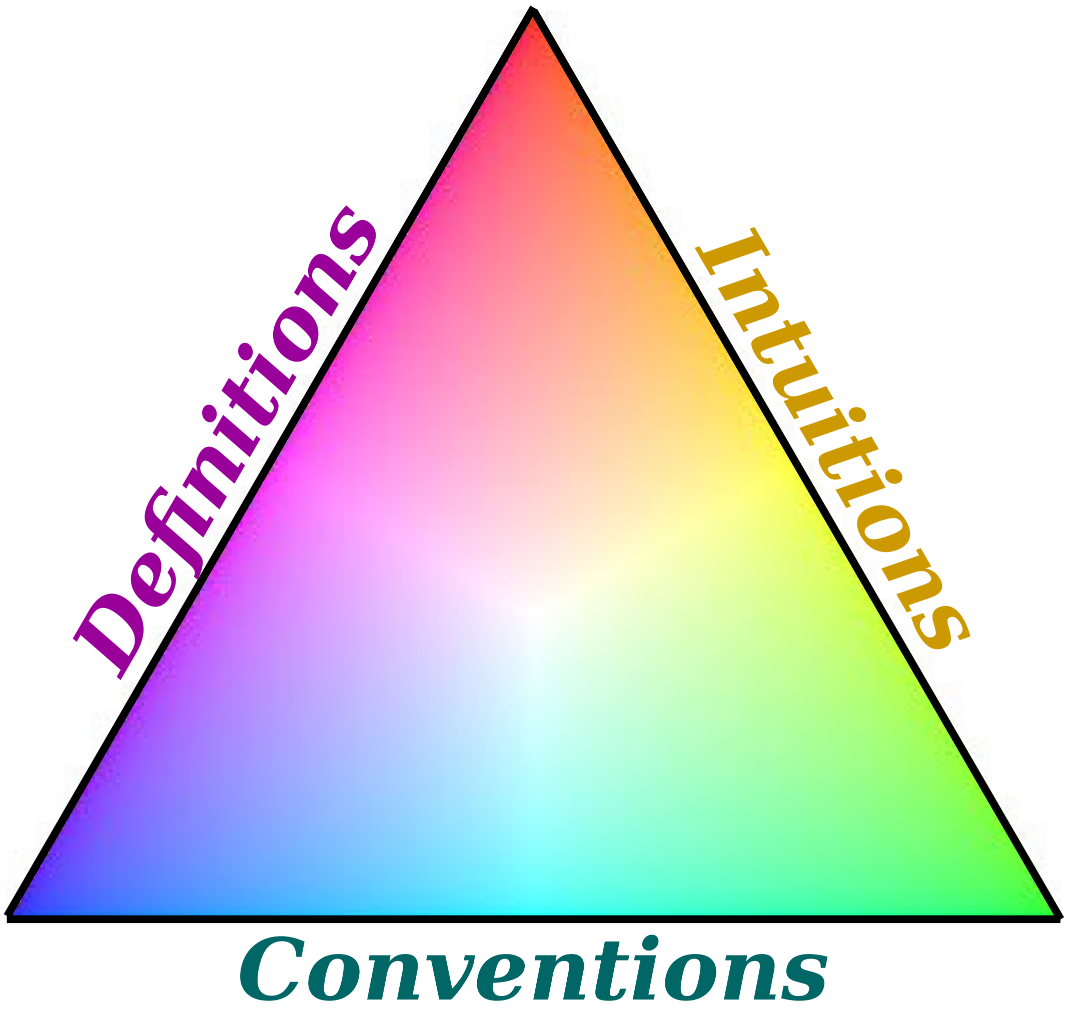
In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required. ■

To Recap



Writing a good proof requires a blend of definitions, intuitions, and conventions.

An integer n is ***even*** if there is an integer k where $n = 2k$.

An integer n is ***odd*** if there is an integer k where $n = 2k+1$.

Definitions tell us what we need to do in a proof.
Many proofs directly reference these definitions.

Let's Draw Some Pictures!

Let's Do Some Math!

Let's Try Some Examples!

Building intuition for results requires creativity, trial, and error.

- Prove universal statements by making arbitrary choices.
- Prove existential statements by making concrete choices.
- Prove "If P, then Q" by assuming P and proving Q.
- Write in complete sentences.
- Number sub-formulas when referring to them.
- Summarize what was shown in proofs by cases.
- Articulate your start and end points.

Mathematical proofs have established conventions that increase rigor and readability.

Your Action Items

- ***Read "Guide to ∈ and ⊆," "Guide to Proofs," and "Guide to Partners."***
  - There's a lot of goodies in there.
- ***Finish and submit Problem Set 0.***
  - Don't put this off until the last minute!
- ***(Optionally) Fill out the Problem Set Matchmaker form.***
  - Want us to connect you with someone else? This is a great way to get started.

Next Time

- ***Indirect Proofs***
  - How do you prove something without actually proving it?
- ***Mathematical Implications***
  - What exactly does "if *P*, then *Q*" mean?
- ***Proof by Contrapositive***
  - A helpful technique for proving implications.
- ***Proof by Contradiction***
  - Proving something is true by showing it can't be false.

Appendix: ***Proofs as Dialogs***

Proofs as a Dialog

Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

Now, let $z = k - 34$.

Proofs as a Dialog

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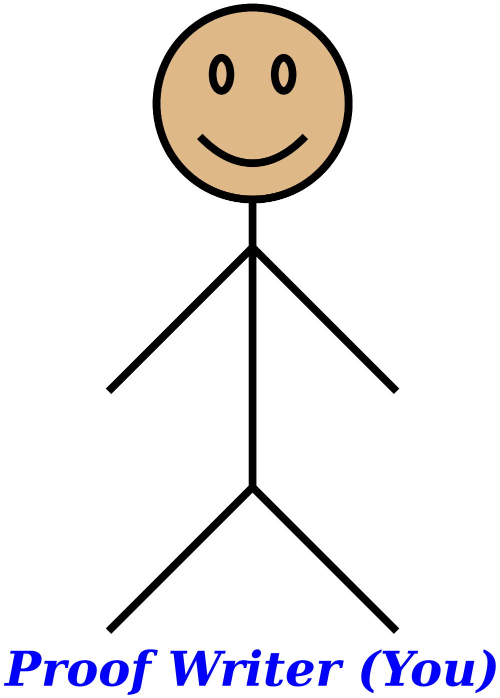

Proofs as a Dialog

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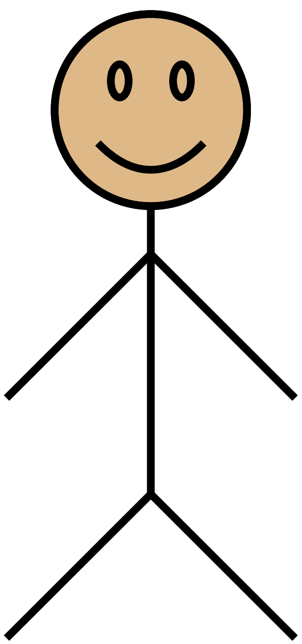
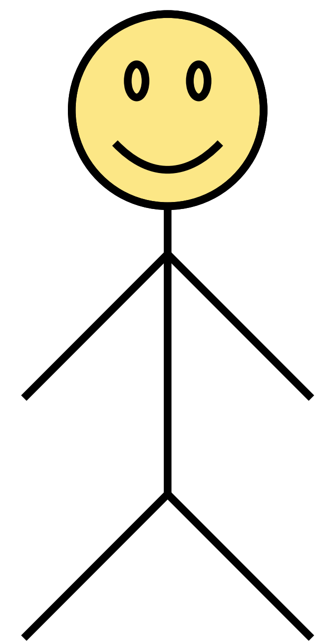


Proof Writer (You)



Proof Reader

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Proofs as a Dialog

Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

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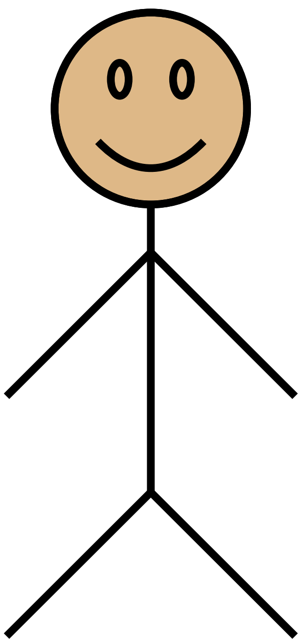
$n = 137$

Reader *Picks*



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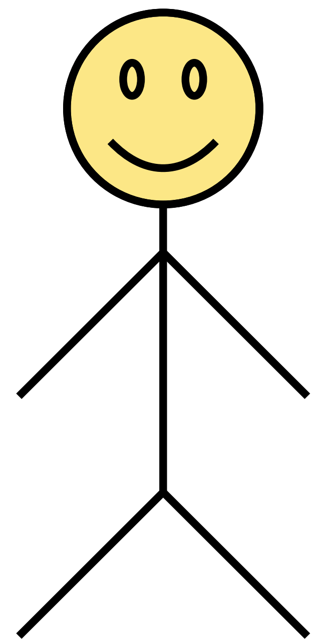

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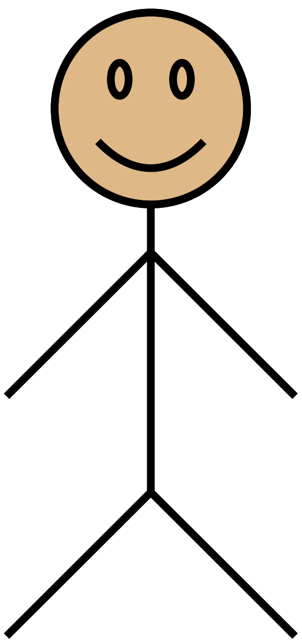

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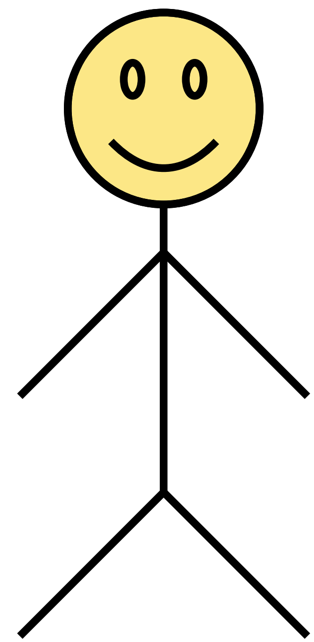

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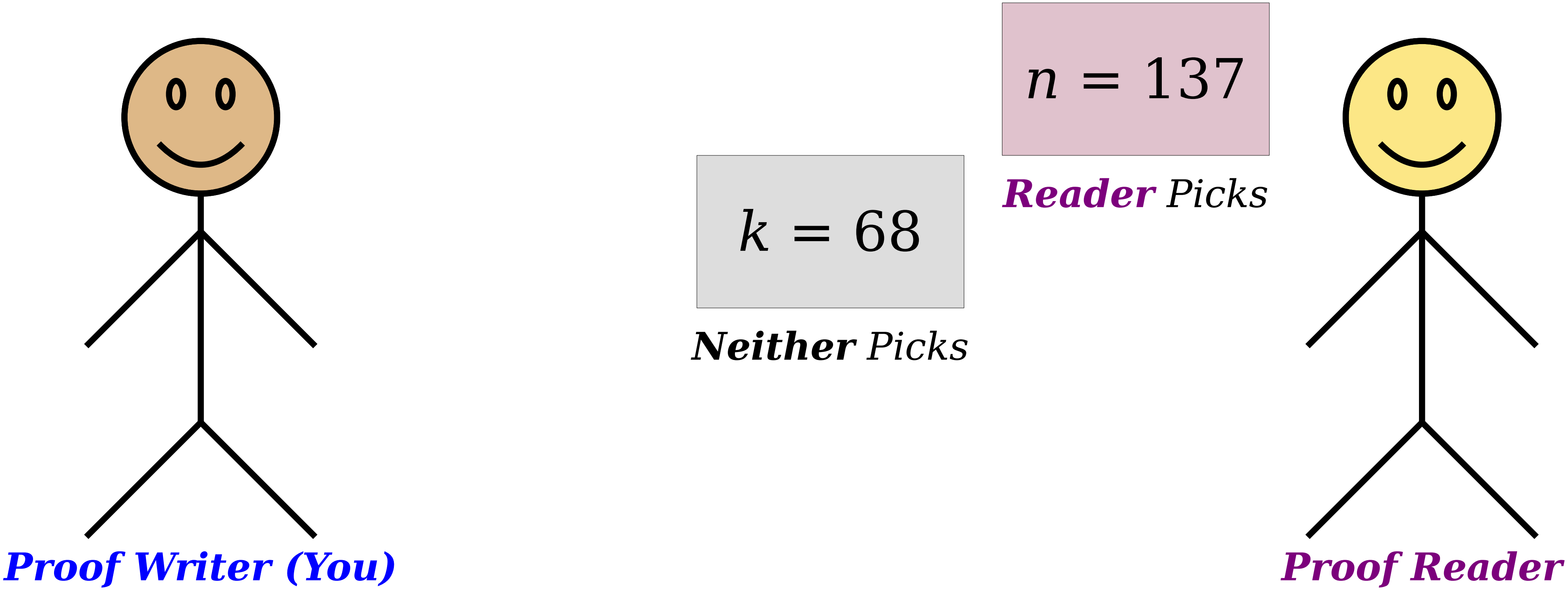

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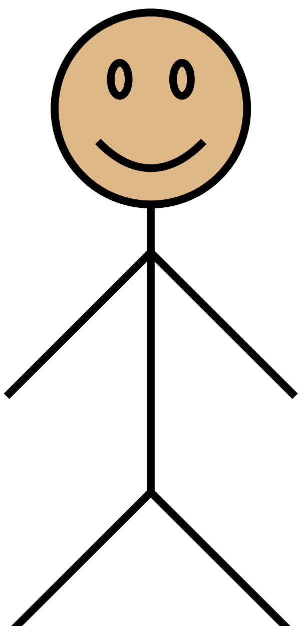


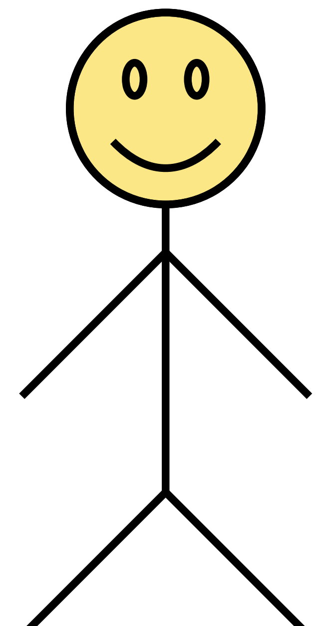
Proof Writer (You)

$k = 68$

Neither *Picks*

$n = 137$

Reader *Picks*



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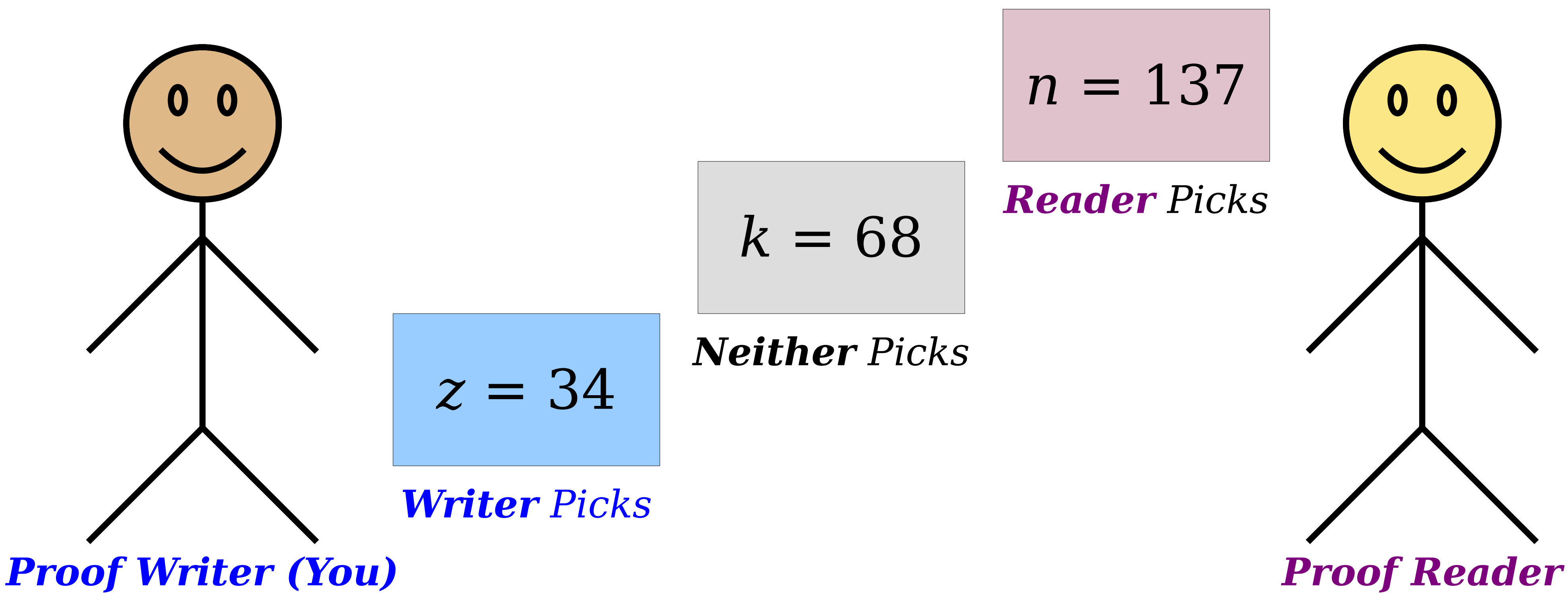

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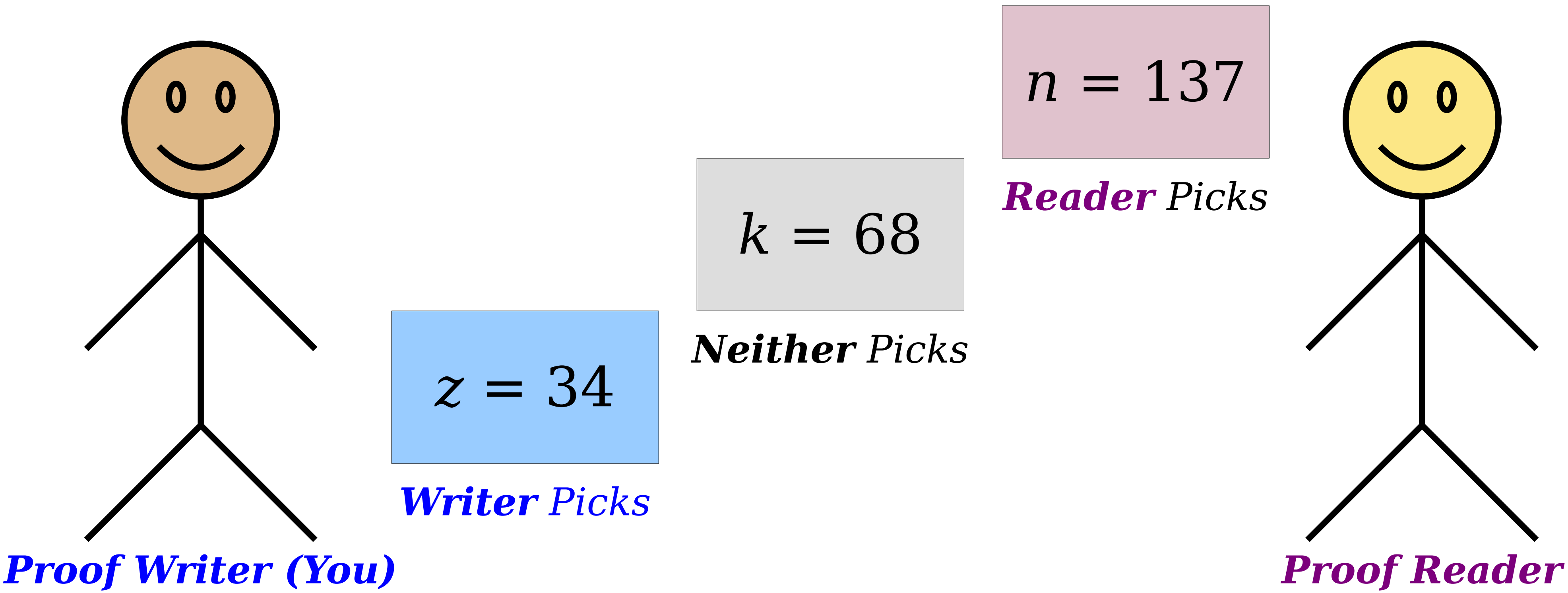

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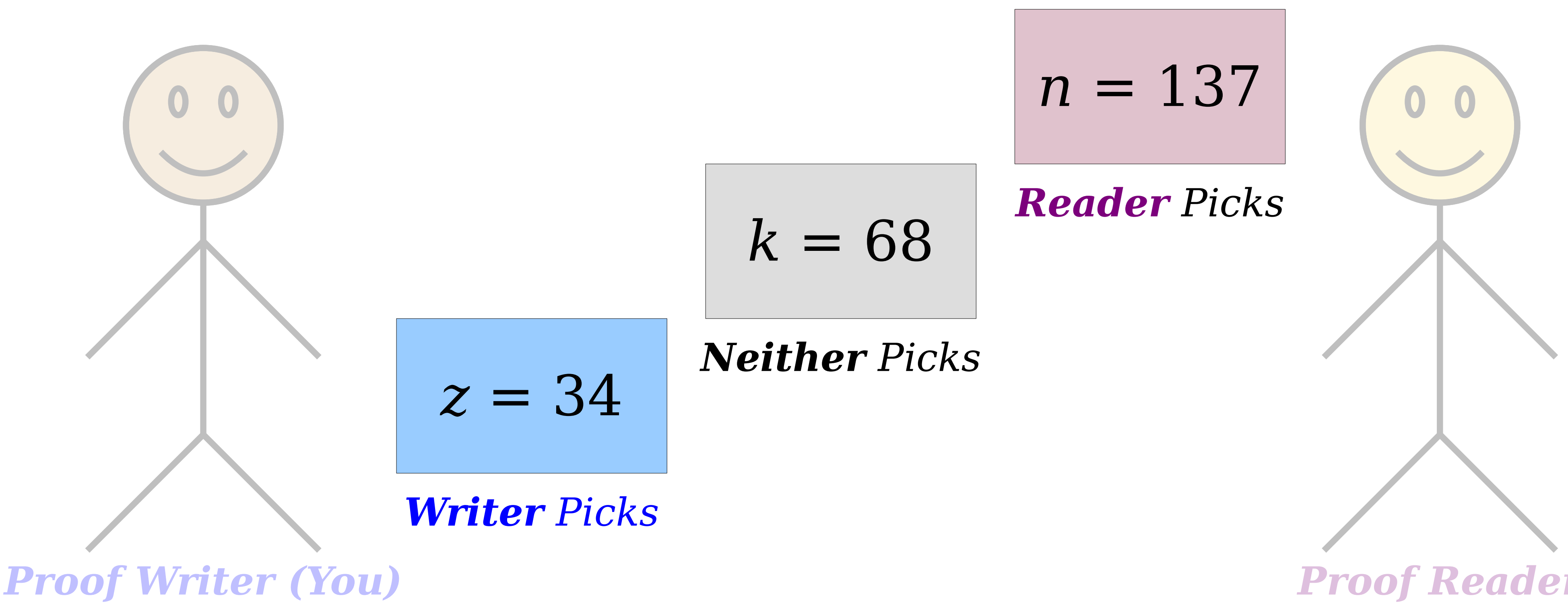
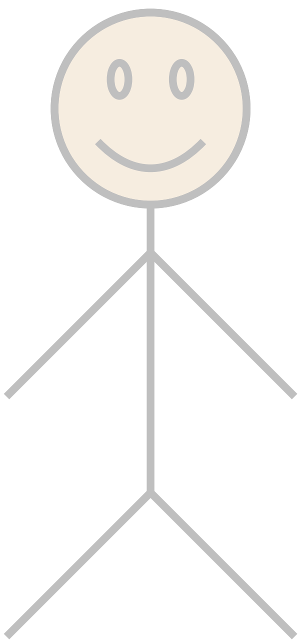


Each of these variables has a distinct, assigned value.

Each variable was either picked by the reader, picked by the writer, or has a value that can be determined from other variables.

Since … r

Now, let $z = k - 34$.



$n = 137$

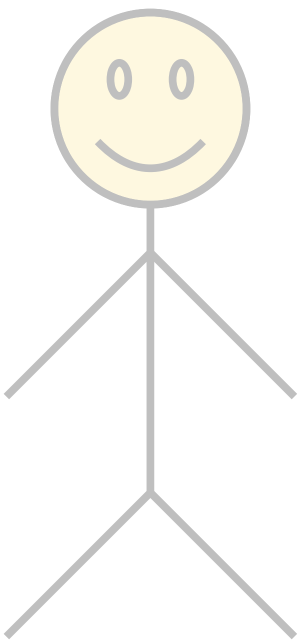
Reader *Picks*

$k = 68$

Neither *Picks*

$z = 34$

Writer *Picks*



Proof Writer (You)

Proof Reader

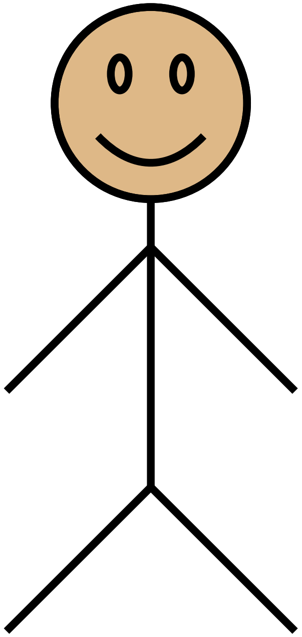
Who Owns What?

- The ***reader*** chooses and owns a value if you use wording like this:
  - Pick a natural number n.
  - Consider some $n \in \mathbb{N}$.
  - Fix a natural number n.
  - Let n be a natural number.
- The ***writer*** (you) chooses and owns a value if you use wording like this:
  - Let $r = n + 1$.
  - Pick $s = n$.
- ***Neither*** of you chooses a value if you use wording like this:
  - Since n is even, we know there is some $k \in \mathbb{Z}$ where $n = 2k$.
  - Because n is odd, there must be some integer k where $n = 2k + 1$.

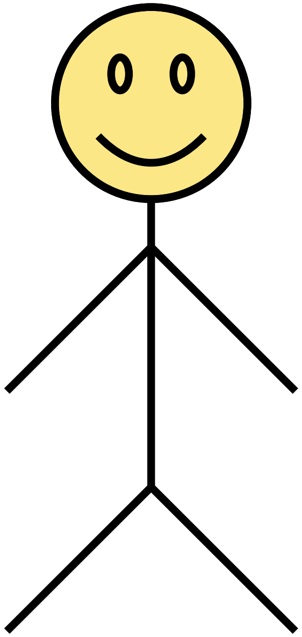
Proofs as a Dialog

Let x be an arbitrary even integer.

Then for any even x, we know that $x+1$ is odd.



Proof Writer (You)



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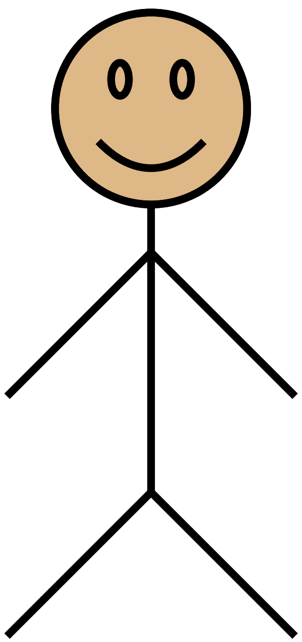

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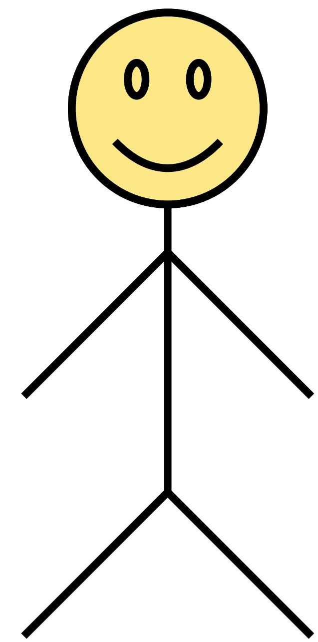

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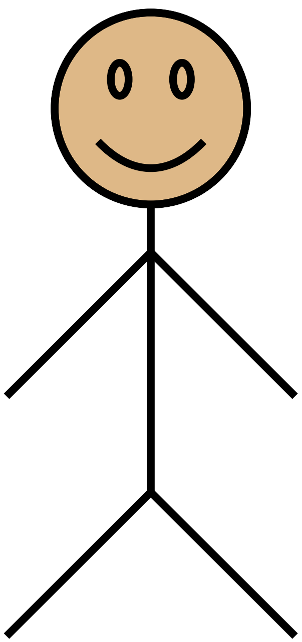
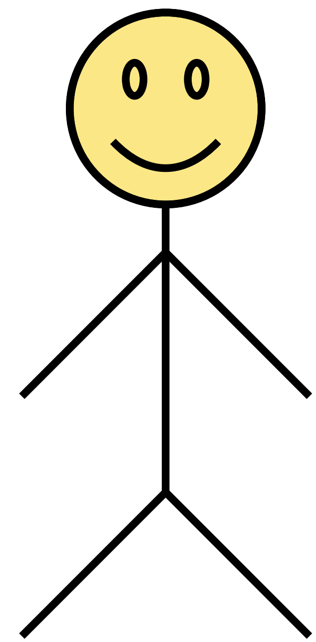


Proof Writer (You)

$x = 242$

Reader *Picks*



Proof Reader

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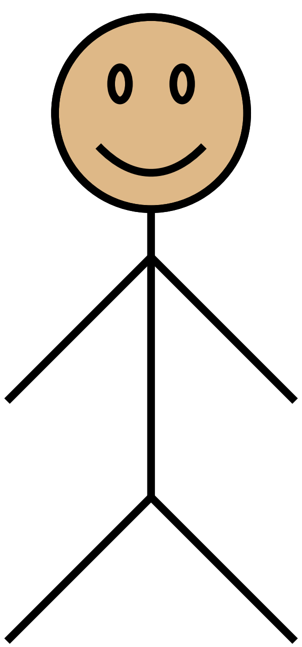

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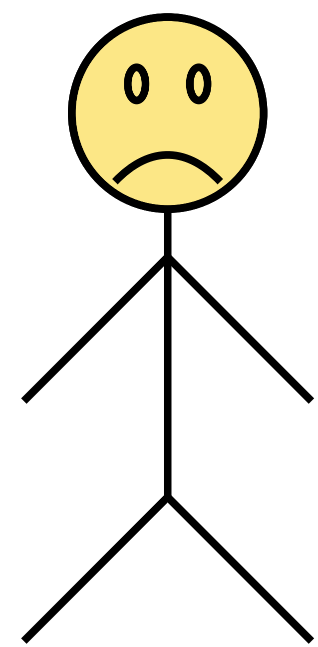

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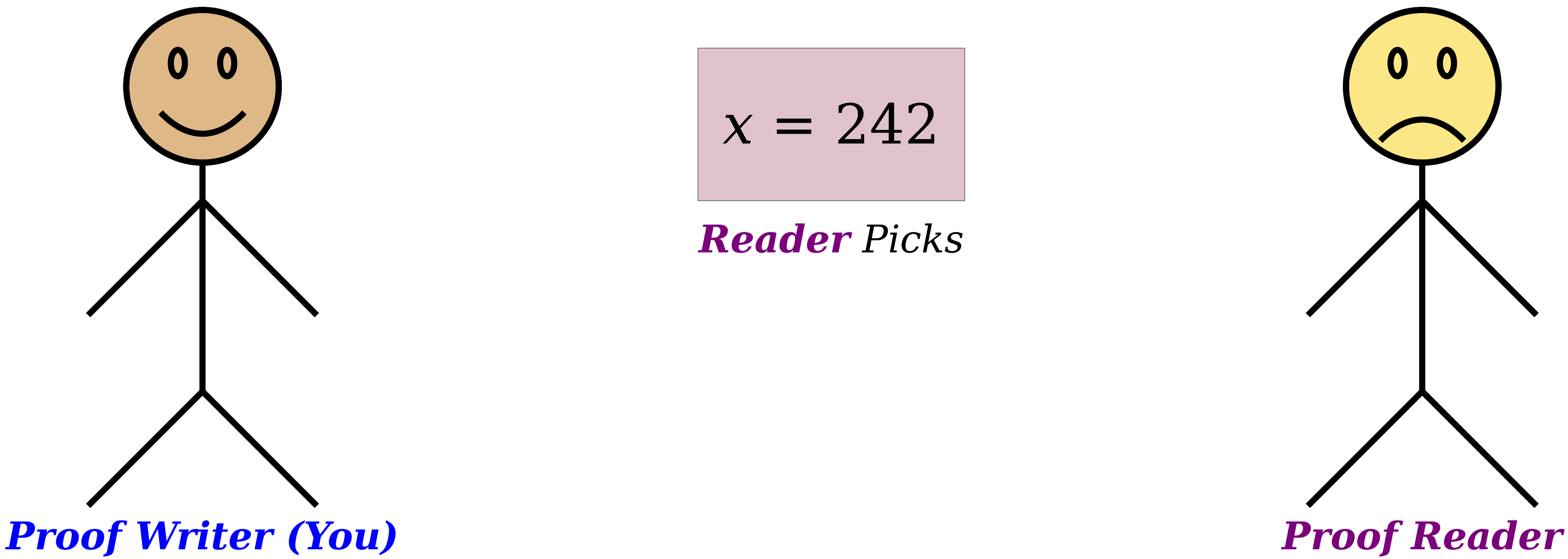

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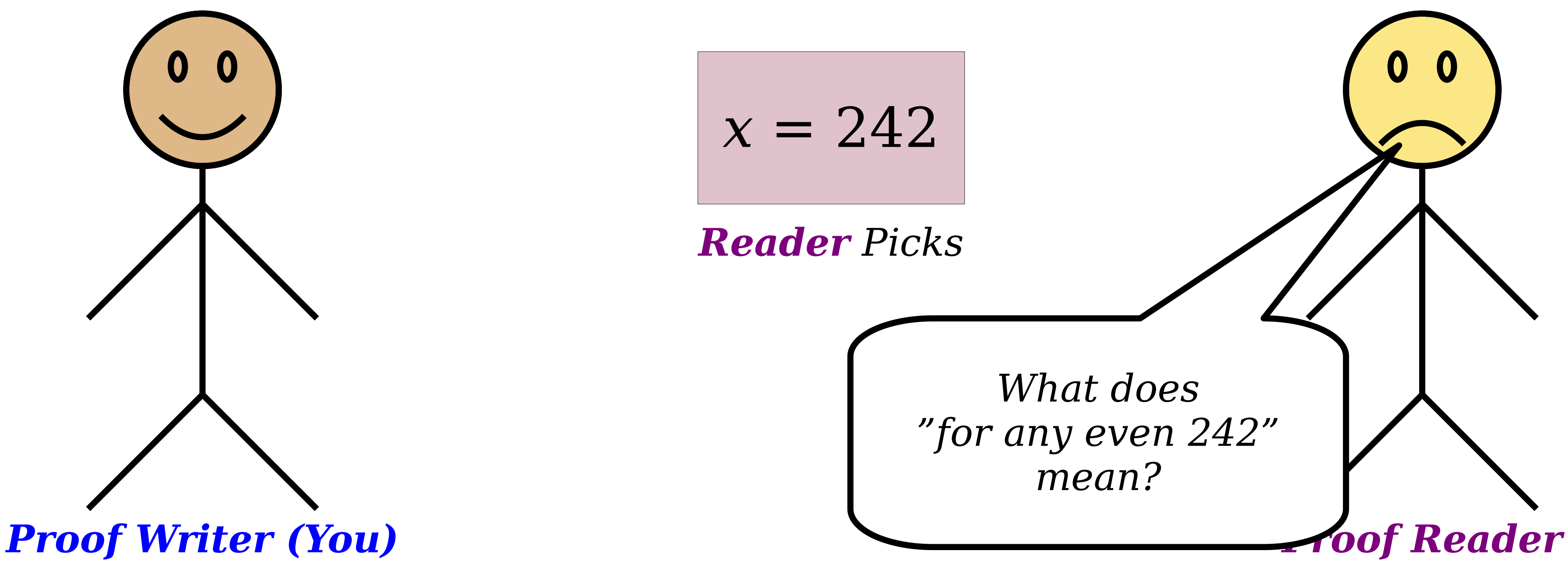

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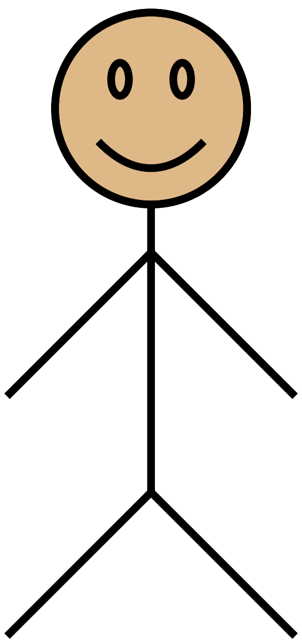

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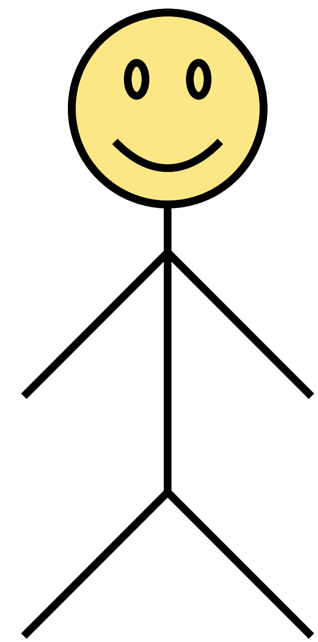

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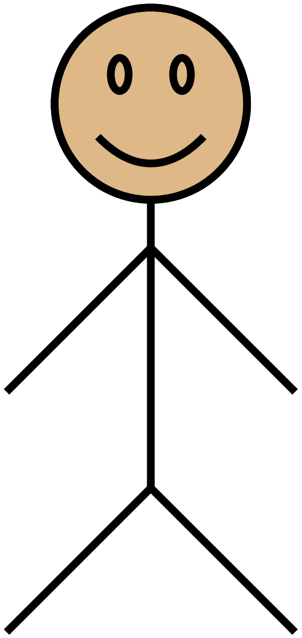
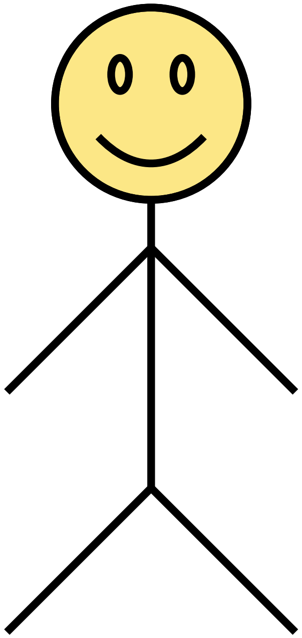


Proof Writer (You)

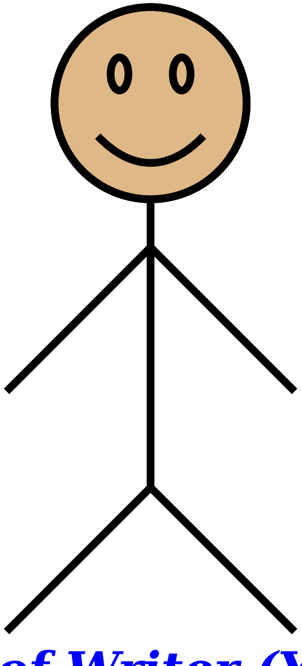


Proof Reader

Proofs as a Dialog

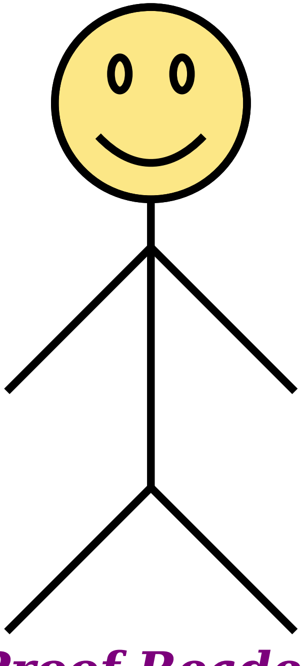
Let x be an arbitrary even integer.

Since x is even, we know that $x+1$ is odd.



$x = 242$

Reader *Picks*



Proof Writer (You)

Proof Reader

Proofs as a Dialog

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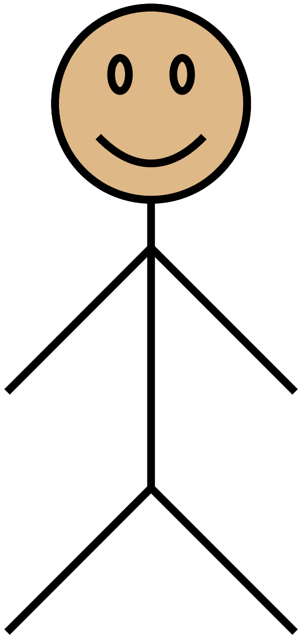
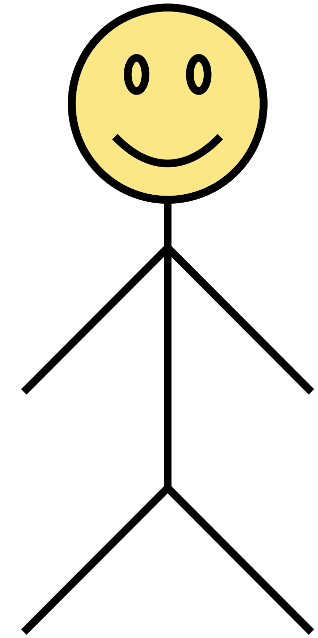


Proof Writer (You)

$x = 242$

Reader *Picks*



Proof Reader

Every variable needs a value.

Avoid talking about "all x" or "every x" when manipulating something concrete.

To prove something is true for any choice of a value for x, let the reader pick x.

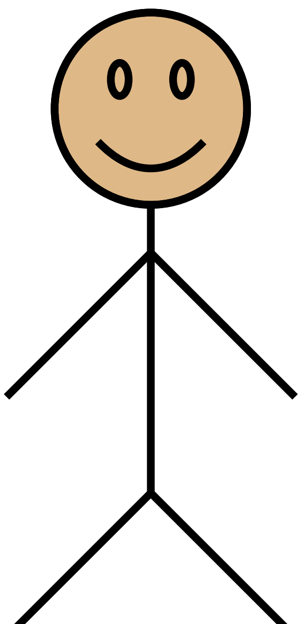
Once you've said something like

Let x be an integer.
Consider an arbitrary $x \in \mathbb{Z}$.
Pick any x.

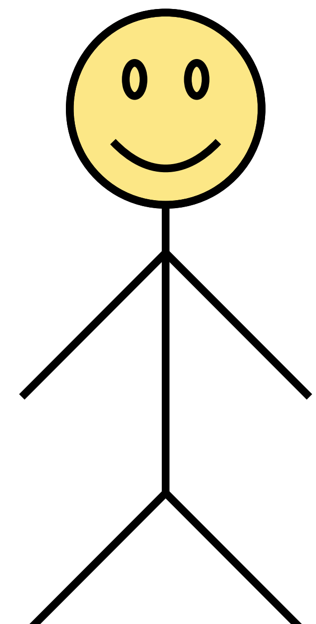
Do not say things like the following:

This means that ***for any*** $x \in \mathbb{Z}$ …
So ***for all*** $x \in \mathbb{Z}$ …

Proofs as a Dialog



Proof Writer (You)

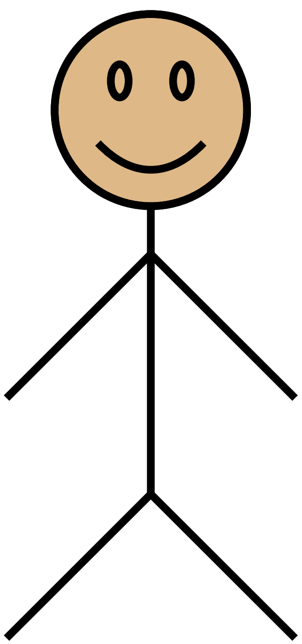


Proof Reader

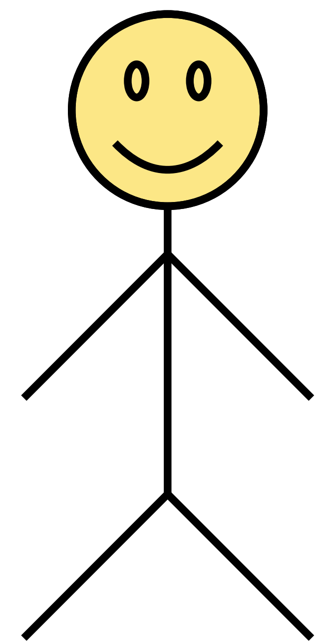
Proofs as a Dialog

Pick two integers m and n where $m+n$ is odd.

Let $n = 1$, which means that $m+1$ is odd.



Proof Writer (You)



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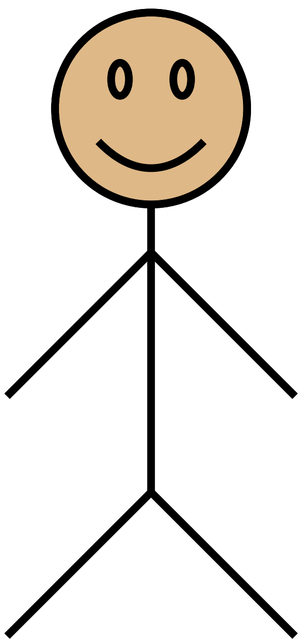
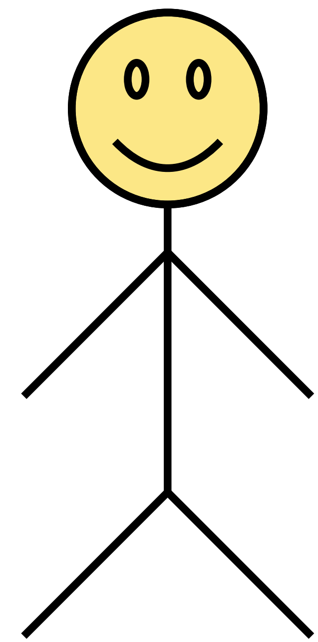


Proof Writer (You)



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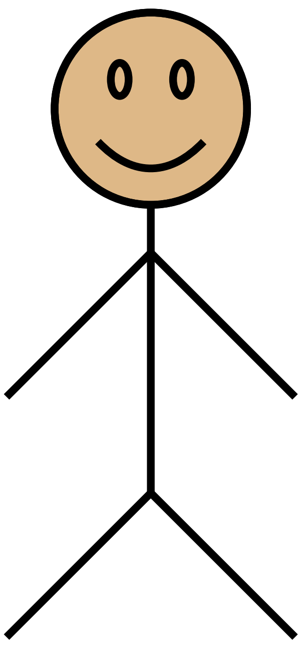


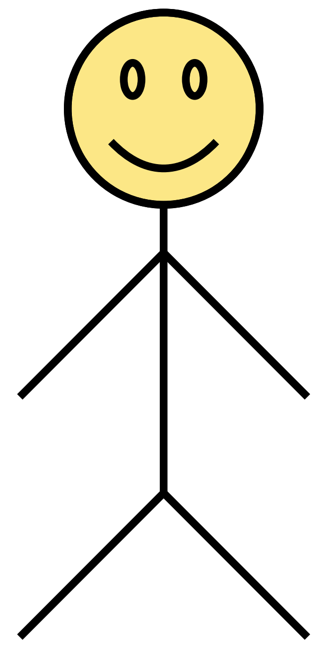
Proof Writer (You)

$m = 103$

Reader *Picks*

$n = 166$

Reader *Picks*



Proof Reader

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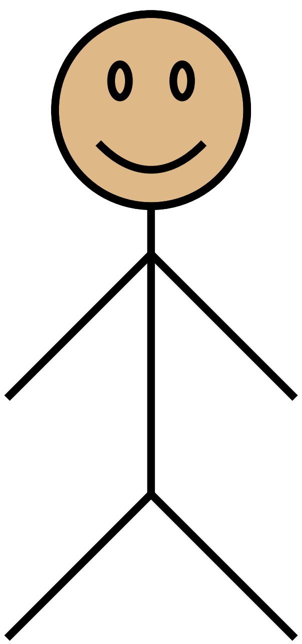
Let $n = 1$, which means that $m+1$ is odd.

$m = 103$

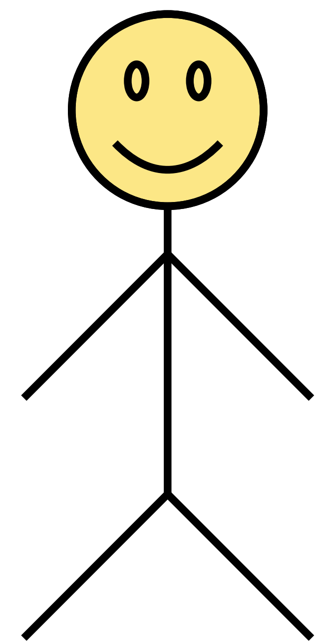
Reader *Picks*

$n = 166$

Reader *Picks*



Proof Writer (You)



Proof Reader

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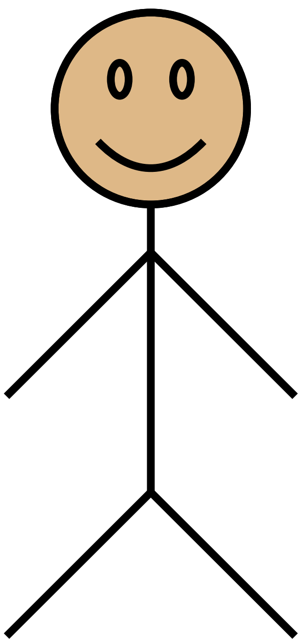

Proof Writer (You)

$m = 103$

Reader *Picks*

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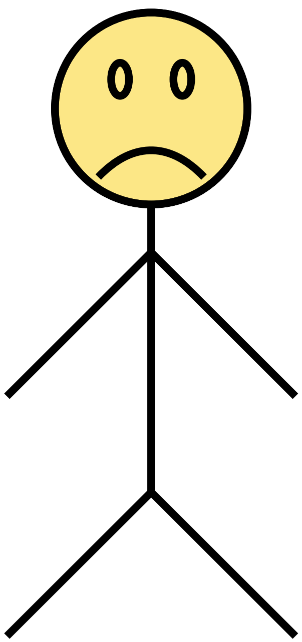
Reader *Picks*



Proof Reader

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Reader *Picks*

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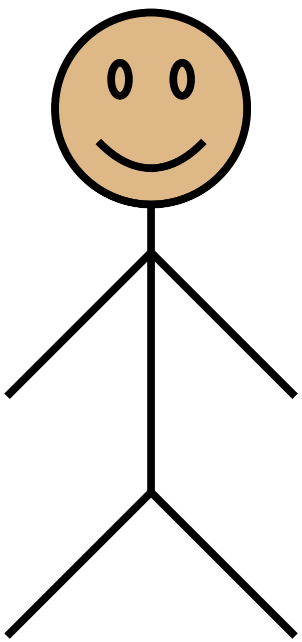
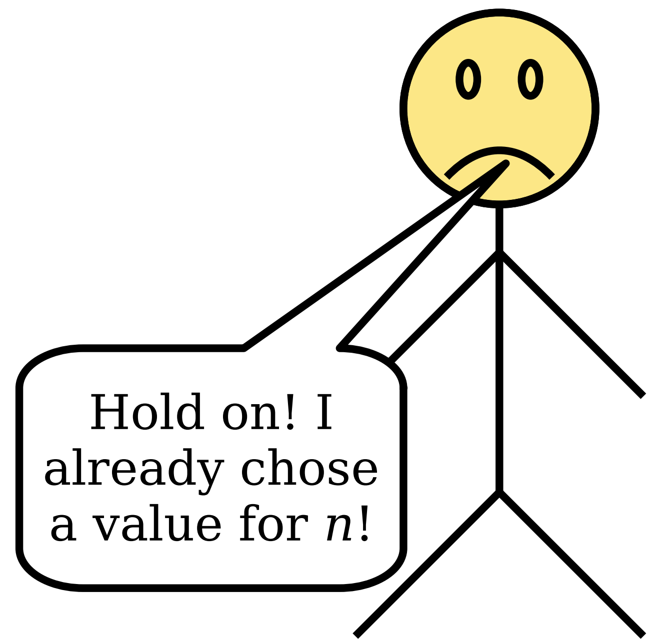
Reader *Picks*



Proof Writer (You)

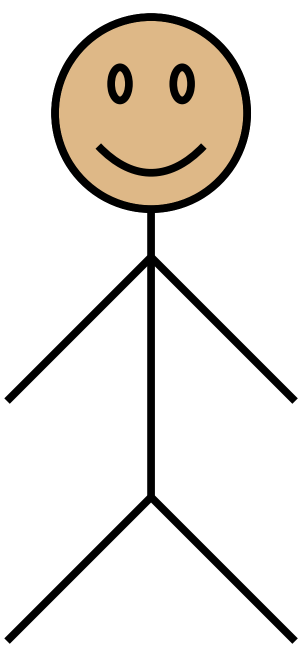


Proof Reader

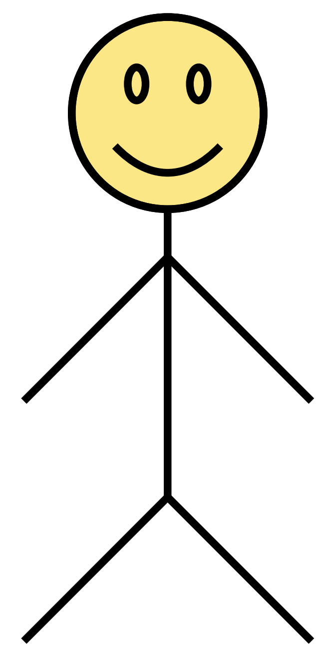
Proofs as a Dialog

Let n = 1.

Pick any integer m where m+1 is odd.



Proof Writer (You)

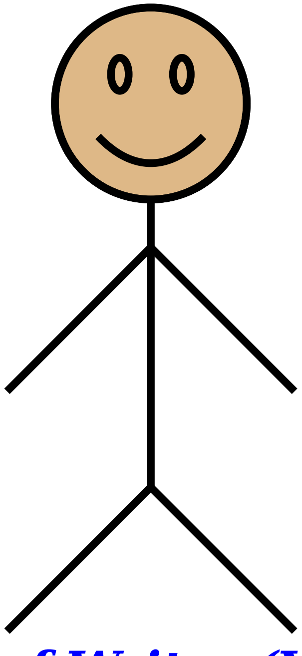


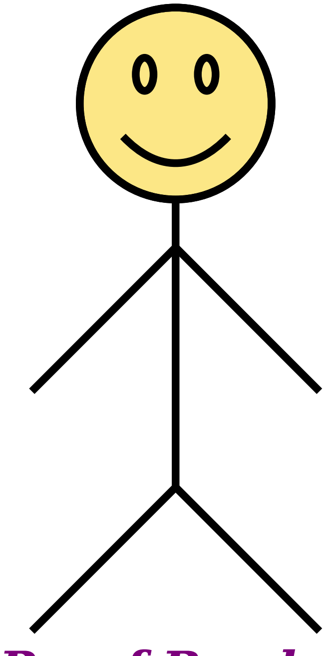
Proof Reader

Proofs as a Dialog

Let $n = 1$.

Pick any integer m where $m+1$ is odd.





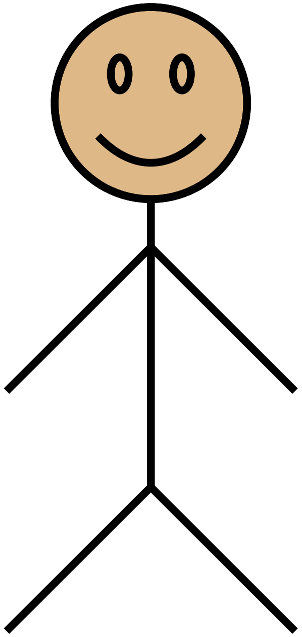
Proofs as a Dialog

Let $n = 1$.

Pick any integer m where $m+1$ is odd.

$n = 1$

Writer *Picks*



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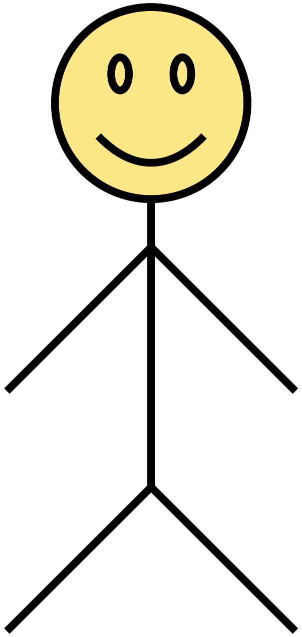

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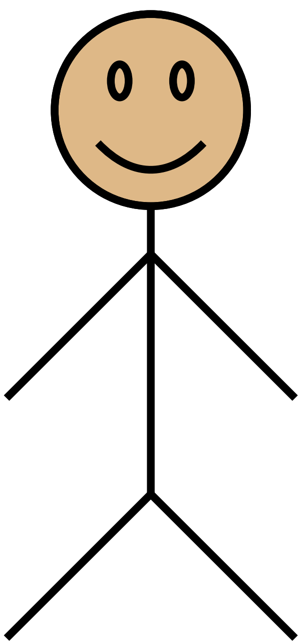
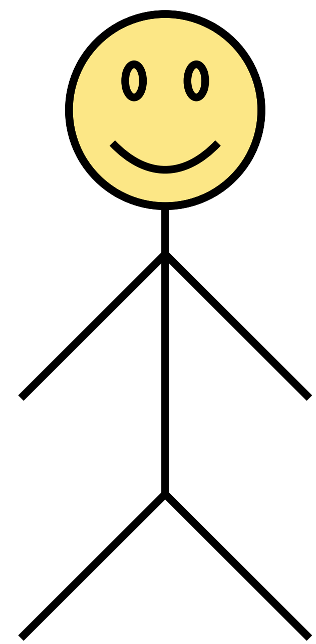


Proof Writer (You)

$n = 1$

Writer *Picks*

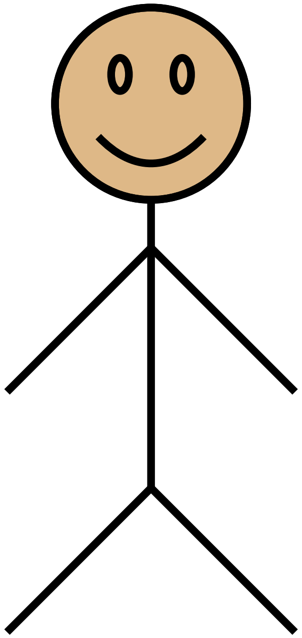


Proof Reader

Proofs as a Dialog

Let $n = 1$.

Pick any integer m where $m+1$ is odd.



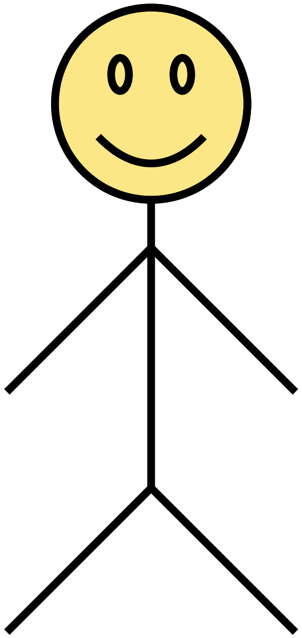
Proof Writer (You)

$m = 166$

Reader *Picks*

$n = 1$

Writer *Picks*



Proof Reader

Proofs as a Dialog

Let n = 1.

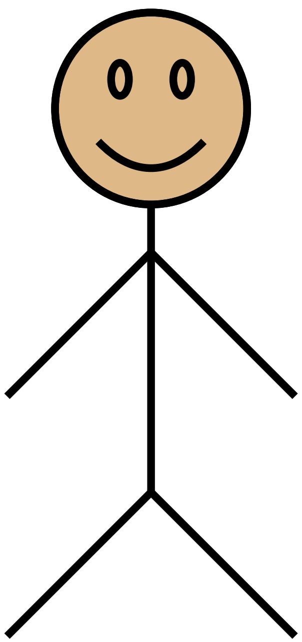
Pick any integer m where m+1 is odd.

m = 166

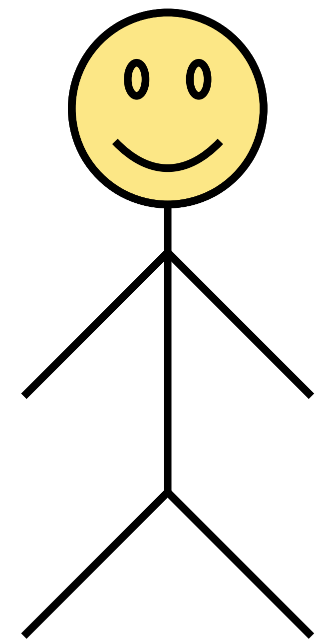
Reader *Picks*

n = 1

Writer *Picks*



Proof Writer (You)



Proof Reader

Proofs as a Dialog

Let $n = 1$.

Pick any integer m where $m+1$ is odd.

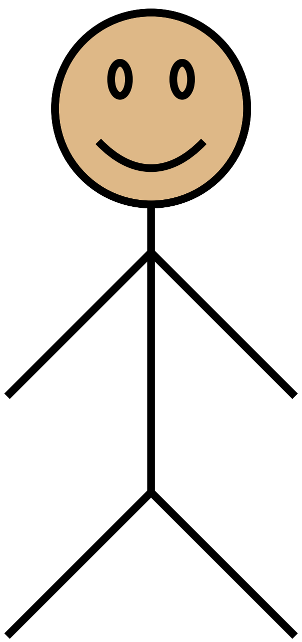
Do we even need n here?

$m = 166$

Reader *Picks*

$n = 1$

Writer *Picks*



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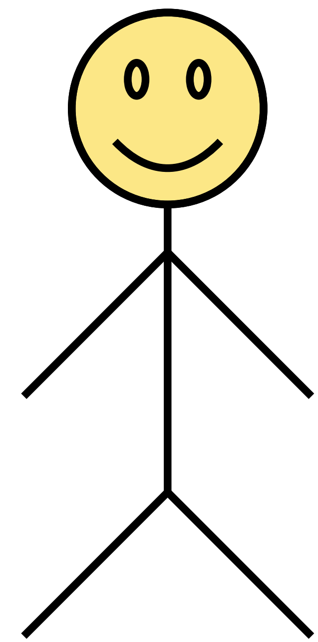

Proof Reader

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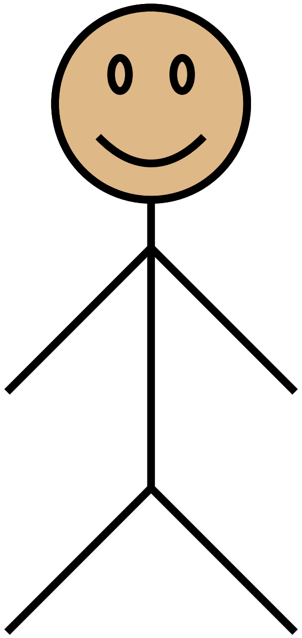

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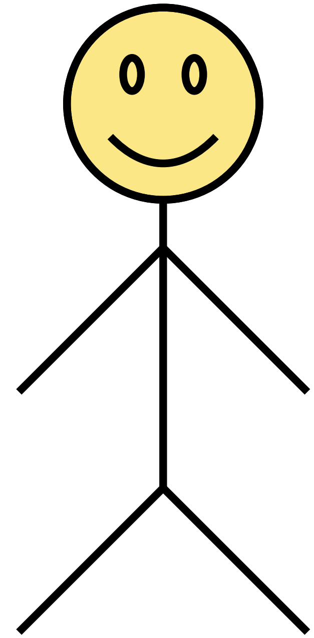

Proof Reader

Proofs as a Dialog

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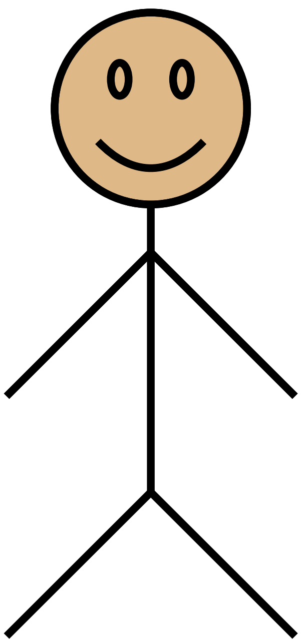
$m = 166$

Reader *Picks*



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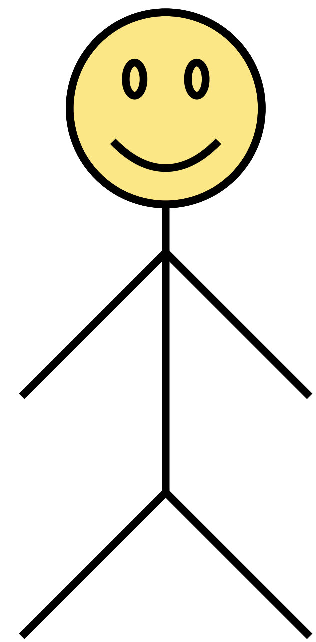

Proof Reader

Be mindful of who owns what variable.

Don’t change something you don’t own.

You don’t always need to name things, especially if they already have a name.